Doctoral Thesis

On Holditch's theorem and related kinematics

David Rochera Plata

Facultat de Ciències Matemàtiques
Dept. de Matemàtiques

Universitat de València

Supervised by:
Juan Monterde García-Pozuelo
José Vicente Beltrán Solsona

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

September 2019

I declare that this thesis titled *On Holditch's theorem and related kinematics* and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.

Valencia, September 2019

David Rochera

We declare that this dissertation presented by David Rochera Plata titled *On Holditch's theorem and related kinematics* has been done under our supervision at the University of Valencia. We also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of Doctor in Mathematics.

Valencia, September 2019

Juan Monterde **José Vicente Beltrán**

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To Maryam.

Acknowledgements

"Where should I go?"
"That depends on where you want to end up."

Lewis Carroll

I've been waiting for this moment some years and I've also been thinking of what to say upon arriving it. Those thoughts faded over time and it forces me to improvise these words again now. I'll try.

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On Holditch's theorem and related kinematics

David Rochera

Abstract

Holditch's theorem is a classical result on areas of planar curves generated by moving chords. The construction is closely related to other kinds of curves such as parallel curves, constant width curves or bicycle curves. The basic properties of these curves are compiled and a historical review on Holditch's theorem and related theorems in kinematics is given. First, the Holditch planar setting is rigorously defined and problems such as the existence of that construction or the avoidance of retrograde movements of the moving chord are considered. In the statement of Holditch's theorem, the area of a hidden ellipse appears. A polygonal approach to the theorem is used to show geometrically where this ellipse comes from. Moreover, immediate generalizations of Holditch's theorem and related results to other contexts are possible. So, in the second part, an introduction to non-Euclidean geometry is given and the extension of such results to constant curvature surfaces is presented. In addition, hidden closed curves in the constant curvature manifold related to the generalized statement of Holditch's theorem are found. Finally, a new extension of Holditch's theorem to space curves is derived in a natural way leading to the concept of Holditch surface.

Introduction

The framework of this thesis is on the study of curves and surfaces generated by moving chords in some spaces. Kinematics studies of moving bars can be found in very ancient books and papers. Among these old results, we have Steiner's formulae for parallel curves, Barbier's theorem for constant width curves and Holditch's theorem.

Holditch's theorem is the most important kernel of this work. Suppose a chord of constant length ℓ moves smoothly always having its endpoints on a planar closed curve giving a full revolution. A fixed point in the chord at a distance p from one end and q from the other (i.e., $\ell = p + q$) will generate another closed curve, namely, the Holditch curve. Holditch's theorem states that the difference between the areas of the initial curve and its Holditch curve is always equal to $\pi\, p\, q$ (see Figure 1). This area is referred to as the Holditch area and note that it coincides with the area of an ellipse with semi-axes p and q, although there are no ellipses in the statement.

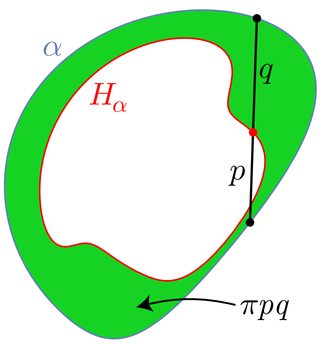

Figure 1: The difference between the areas of the initial curve α and its Holditch curve H_α generated by a chord length $p + q$ is constant and equal to $\pi\, p\, q$.

Holditch's theorem has been in the scope of many mathematicians over time. In fact, some recent references on Holditch's theorem can be found and we are even aware that there are people working on this and related topics.

The *main objectives* of this thesis are:

- To provide a comprehensive historical review on Holditch's theorem and related kinematics results in current language.

- To set the basic definitions on Holditch curves in different settings (such as in the plane, in non-Euclidean geometry or in the space).

- To review and give new constructions and proofs for some results, including those of Steiner, Barbier and Holditch; both in the plane and in non-Euclidean geometry.

- To discover from which ellipse the Holditch area comes and to understand that fact geometrically.

- To generalize Holditch's theorem for space curves following the same natural construction as done in the plane.

It should also be said that we tried to make this thesis as self-contained as possible, assuming just a basis in some differential geometry concepts. This justifies the length of the present work, where Chapters 1, 2 and 4 constitute introductory background reviews or constructions.

The most important results to accomplish the objectives above are outlined next in a chapter by chapter display.

Chapter 1

In the first part, Section 1.1, a brief introduction to some basic concepts of differential geometry are recalled with the aim to establish the notation to be followed throughout the work.

In the second part of the chapter, some kinds of classical plane curves are defined and some of their basic properties are given:

- Section 1.2 is about *parallel curves*. Some of their basic properties are recalled and in Theorem 1.8 a version of Steiner's formulae for the area and the length of these curves is stated and proved. Later, the not so well-known notion of *parallel curves according to an angle* are defined as an extension of the classical parallel curves. In Theorem 1.11 we give a sufficient condition for the regularity of these kinds of curves. The extension of Steiner's formulae in this context is also possible.

- Section 1.3 is on *constant width curves*. A brief introduction to supporting lines, widths and support functions for convex curves is presented. Barbier's theorem is recalled in this context.

- Section 1.4 is about the common definition of the *bicycle tire-tracks curves* given by the usual and simple model of a bicycle as a moving segment.

Chapter 2

The main objective of this chapter is to compile some of the ancient and forgotten work on Holditch's theorem and related kinematics. Most of the references on this subject date from more than 130 years ago.

The history of Holditch's theorem is given and different proofs of it are given. Later, extensions of the classical statement are reviewed, such as the generalizations due to Woolhouse, Leudesdorf and Elliott. With regard to some of these extensions, related results on kinematics are given in Section 2.3, such as Kempe's theorem.

Among the generalizations proposed by Elliott, results on volumes and surface areas are considered. In the last part of the chapter, Section 2.5, partial results on spherical kinematics also due to Elliott are studied.

Chapter 3

In this chapter, most of the main results in the plane involving moving chords and Holditch's theorem are given. Some parts of this chapter, especially Sections 3.3 and 3.4, are based on the work [65]. Nevertheless, a more detailed approach is followed here.

First, basic concepts such as Holditch functions, Holditch curves and retrograde motion in the plane are defined rigorously. Theorem 3.6 on the existence of Holditch curves is an original result.

In Section 3.1.3, the definition of some important angles that appear in a Holditch setting is given. These angles are useful in next section, Section 3.1.4, to find a new characterization for retrograde motion in strictly convex curves (Theorem 3.17). For any closed curve (not necessarily convex), a sufficient condition is given with the definition of Holditch radius (Theorem 3.14).

In Section 3.2 some properties in a Holditch setting are presented. It is seen that:

- The angle function which determines the direction of the moving chord is increasing for a strictly convex curve (Proposition 3.22).
- Holditch curves of regular curves are regular for a chord length less than the Holditch radius (Theorem 3.25).
- Holditch curves of a constant width curve for a chord length equal to such width are parallel curves to the initial one (Theorem 3.28). In particular, they are of constant width.
- Barbier's theorem can be proved as a limiting case of Holditch's theorem.

Section 3.3 deals with the definition and the continuity of the Holditch map, the function which sends any simple closed $\mathcal{C}^1$-curve into its Holditch curve for a chord length. The main contributions here are Theorem 3.33 and Corollary 3.34, which ensure the uniform continuity of the Holditch map for both smooth and piecewise-smooth initial curves.

With the continuity of the Holditch map, in Section 3.4 the ellipse which appears in the statement of Holditch's theorem is unveiled via polygonal approximations to the initial curve. The main fact is the area preservation in each step of a convergent sequence to the smooth case. This area preservation is proved in two different ways. In one of them, the main idea consists of the use of shear transformations. By the approach of this section, polygonal versions of Holditch's theorem are also given (Proposition 3.41 and Theorem 3.43).

Finally, in the last part, Section 3.5, a more general setting to describe curves from an initial one in the plane is given. These curves are referred to by the name of *generated curves*. Lemma 3.50 is the most important result of this section. From this general result, the classical cited theorems due to Steiner, Barbier and Holditch are derived. Moreover, a result on areas swept out by bicycle tire-tracks is also deduced. This last section presents the planar case as a preliminary approach to the generalization given in next chapters.

Chapter 4

This chapter constitutes the foundations of the models of non-Euclidean geometry which are used in the next chapter: the sphere and the two-sheeted hyperboloid. The non-Euclidean space M^K is presented as embedded in $\mathbb{R}^3$ as a quadratic space. It is seen that M^K is, in fact, a Riemannian manifold with the inherited metric of the quadratic space and has constant Gauss curvature K. Some basic properties and definitions on M^K are given regarding normal vectors, orthonormal frames, angles, geodesics and areas.

In the last part of this chapter, the generalized versions of parallel curves (with their extension according to an angle), constant width curves and bicycle curves are presented.

We think that the approach to M^K given in this chapter is interesting mainly due to two reasons. On the one hand, this machinery allows us to work with the two models of non-Euclidean geometry (the sphere and the hyperboloid) at the same time with the same techniques. On the other hand, no reference has been found where this construction is done in such a detailed form.

Chapter 5

This chapter deals with moving chords in a 2-dimensional constant curvature manifold M^K. Some parts of this chapter are detailed and expanded versions of the work [64].

First of all, a brief introduction to the naturally defined Jacobi fields along the moving chord is given both in the plane and in a non-zero constant curvature manifold.

Later, the generalized notion of *generated curves* of Chapter 3 is considered. In this part, regularity results on generated curves are found, a generalization of the envelope theorem is given and an orthonormal frame adapted to generated curves is defined.

The most important results of this work involving moving chords in M^K are derived in Section 5.3.4. These results are the generalized versions to constant curvature surfaces of Steiner's formulae for parallel curves, Barbier's theorem for constant width curves, Holditch's theorem and the swept out area of bicycle curves. These theorems are proved with a new approach thanks to Lemma 5.12, which relates the geodesic curvatures of the generated curve and the initial one. This part is closely related to some old works due to Vidal Abascal and Santaló.

Finally, in Section 5.4 some other points on Holditch's theorem are considered:

- The definition of Holditch curve in M^K and retrograde motion.
- The relation between swept out areas by the moving chord in M^K and statements on a difference of areas. Finally, a version of Holditch's theorem for convex curves is proved by this approach (Corollary 5.34).
- The realization of the Holditch constant of the generalized statement as the area of a closed curve in M^K (Proposition 5.35 and Theorem 5.36). A relation with cruciform curves is also found (Proposition 5.38).
- Holditch's and Barbier's formulae characterize the manifold, in the sense that if a Holditch's or Barbier's type formula works, then the Gauss curvature of the manifold is constant.

Chapter 6

This chapter is an extended version of the work [66].

Here, the natural generalization of Holditch's theorem to space curves is done. It begins with the definition of a Holditch surface and some of its properties. The idea to define the Holditch surface came from the work of Arnol'd in singularity theory and wave fronts. With that, the extension of

Holditch's theorem is on areas on the Holditch surface. In Section 6.4 is proved that for planar curves the area in the Holditch surface is planar and it coincides with the Holditch area. This constitutes also a new proof via Holditch surfaces of Holditch's theorem in the plane (Theorem 6.10).

The main results are Theorems 6.11 and 6.14 where Holditch's theorem is generalized with areas in the Holditch surface and an approximation of such an area is given in terms of the curvature and the torsion of the initial curve. Finally, it is shown that the only minimal non-planar regular Holditch surface is the helicoid (Theorem 6.19).

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Chapter 1

Some classical differential geometry

First, a very brief and elementary introduction to some concepts in differential geometry is given. Thus, general considerations on notation are specified. In the second part of this chapter, different kinds of planar curves (such as parallel curves, constant width curves and bicycle curves) are introduced and some of their basic properties on areas and lengths are presented.

1.1 Introduction to differential geometry

Throughout the work, it will be understood that the reader has a basis in classical differential geometry, [20], and in Riemannian geometry, [21]. In this section, some elementary facts will be recalled with the aim of establishing some notation that is going to be followed in other chapters.

1.1.1 Some definitions, notation and results

The derivative of a function $\alpha = \alpha(t)$ with respect to t will be denoted indiscriminately by $\alpha'(t)$ or by $\frac{\mathrm{d}\alpha}{\mathrm{d}t}(t)$.

Definition 1.1 (Regular curve)**.** A parameterized differentiable curve $\alpha : I \to \mathbb{R}^n$, where I is some real interval, is said to be *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

Thus, note that the term *regular parametric curve* assumes differentiability. The term *smooth curve* refers to a curve which is *differentiable* up to a desired order. Sometimes, the term *parametric curve* is used assuming differentiability.

The domain of a parametric curve will be usually denoted by I, which will be taken as a real interval unless stated otherwise.

The trace of a parametric curve α will be denoted by α^*. By abuse of notation, to indicate that two parametric curves α and $\bar{\alpha}$ have the same trace, $\alpha^* = \bar{\alpha}^*$, it is sometimes written $\alpha = \bar{\alpha}$.

Definition 1.2 (Arc-length parameterization)**.** If $\|\alpha'(t)\| = 1$, the curve α is said to be *arc-length* parameterized.

Recall that any regular curve can be reparameterized by arc length.

Planar curves

A parametric curve $\alpha : I \to \mathbb{R}^2$ is said to be *planar*. The *tangent* and *normal vectors* of a planar curve α will be denoted by $\mathbf{t}$ and $\mathbf{n}$, respectively.

Let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $J(a, b) = (-b, a)$. Thus, $\mathbf{n} = J\mathbf{t}$ and $\{\mathbf{t}, \mathbf{n}\}$ is a positively oriented frame in the plane.

Definition 1.3 (Jordan curve)**.** A *Jordan curve* is defined as a *simple closed planar curve.*

A Jordan curve is said to be *positively oriented* if it is traced out counterclockwise.

Let $R \subset \mathbb{R}^2$ be the region encircled by a closed curve α. The *area of R* is defined by

$$\mathcal{A}(\alpha) = \iint_R \mathrm{d}(x, y).$$

Notice that it is always a positive number.

The *signed area* enclosed by a closed curve α, denoted by $\tilde{\mathcal{A}}(\alpha)$, is defined to be $\mathcal{A}(\alpha)$ if α is positively oriented or $-\mathcal{A}(\alpha)$ otherwise.

If $\alpha(s) = \big(x(s),\, y(s)\big)$ is a positively oriented planar closed curve, then thanks to Green's theorem:

$$\mathcal{A}(\alpha) = \iint_R \mathrm{d}(x, y) = \int_\alpha x \, \mathrm{d}y = \int_I x(s)\, y'(s) \, \mathrm{d}s.$$

Equivalently,

$$\mathcal{A}(\alpha) = -\int_I y(s)\, x'(s) \, \mathrm{d}s = \frac{1}{2}\int_I \big(x(s)\, y'(s) - y(s)\, x'(s)\big) \, \mathrm{d}s.$$

If the curve α is negatively oriented, then

$$\mathcal{A}(\phi) = \iint_R \mathrm{d}(x, y) = -\int_\alpha x \, \mathrm{d}y.$$

Therefore, the signed area enclosed by α can be computed by

$$\tilde{\mathcal{A}}(\alpha) = \int_I x(s)\, y'(s) \, \mathrm{d}s.$$

If the curve α is described by a point P, the defined areas may also be written $\mathcal{A}(P)$ or $\tilde{\mathcal{A}}(P)$.

Space curves

Given a space curve $\alpha : I \to \mathbb{R}^3$, its binormal vector will be denoted by $\mathbf{b}$ and its *torsion* by τ. By convention, define the torsion of α such that

$$\tau(s) = \big\langle \mathbf{b}'(s), \mathbf{n}(s) \big\rangle.$$

Therefore, the sign convention is such that $\mathbf{b}'(s) = \tau(s)\, \mathbf{n}(s)$. (The torsion of space curves will be used in Chapter 6.)

Solid volumes

If Ω is a solid volume in $\mathbb{R}^3$, its volume will be denoted by $\mathcal{V}(\Omega)$ and its boundary by $\partial\Omega$. As a consequence of the divergence theorem for a vector field $F(x, y, z) = (0, 0, z)$:

$$\mathcal{V}(\Omega) = \iiint_\Omega \mathrm{d}V = \iint_{\partial\Omega} \big\langle (0, 0, z),\, \mathbf{N}(x, y, z) \big\rangle \, \mathrm{d}S,$$

with $\mathbf{N}$ being the normal vector at each point $(x, y, z) \in \partial\Omega$ and $\mathrm{d}S$ the area element. Now, if

$$f(u, v) = \big(x(u, v), y(u, v), \, z(u, v)\big), \quad (u, v) \in A,$$

is a parameterization of $\partial\Omega$, with A being some domain in $\mathbb{R}^2$, then

$$\mathcal{V}(\Omega) = \iint_A z(u, v) \, \big(x_u(u, v) \, y_v(u, v) - x_v(u, v) \, y_u(u, v)\big) \, \mathrm{d}(u, v).$$

In Cartesian coordinates, the previous expression is equivalent to

$$\mathcal{V}(\Omega) = \iint_{\partial\Omega} z \, dx \, dy.$$

This expression will be used in Section 2.4 to compute some volumes.

1.1.2 General considerations

On closed curves

Throughout this work, $\alpha : I \to M$ will be a parametric curve on M defined on a real interval I, where M will usually be some 2-dimensional manifold embedded in $\mathbb{R}^3$.

If I is closed, we will assume that α is the restriction to I of some parametric curve $\bar{\alpha} : \bar{I} \to M$, $I \subset \bar{I}$ defined in an open interval $\bar{I}$ such that $\alpha(s) = \bar{\alpha}(s)$ for all $s \in I$ and with the same trace. Thus, it makes sense to define the derivative α' of α at any point of I.

A simple closed curve $\alpha : I \to M$ will be understood to be defined to the whole $\mathbb{R}$ (with the same velocity at the same points of the extension) without making explicit mention to it.

On trigonometric functions

The use of the 2-argument arctangent will be appropriate to choose the correct angle in some applications of subsequent chapters.

Definition 1.4 (2-argument arctangent)**.** The 2-*argument arctangent* is the function $\operatorname{atan2} : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \,]-\pi, \pi]$ defined as the argument of the complex number $x + \mathbf{i}\, y$:

$$\operatorname{atan2}(y, x) := \arg(x + \mathbf{i}\, y).$$

That is to say, it gives the unique angle $\theta \in \,]-\pi, \pi]$ such that

$$\begin{cases} x = \cos\theta, \\ y = \sin\theta. \end{cases}$$

If $x > 0$, we have that

$$\operatorname{atan2}(y, x) = \arctan\left(\frac{y}{x}\right).$$

Also, recall the following identities. They will be useful when working with constant curvature manifolds (Chapters 4 and 5):

1. For all $x \in \mathbb{R}$, $\cos(\mathbf{i}\, x) = \cosh(x)$.
2. For all $x \in \mathbb{R}$, $\sin(\mathbf{i}\, x) = \mathbf{i} \sinh(x)$.
3. For all $x \geq 1$, $-\mathbf{i} \arccos(x) = \operatorname{argcosh}(x)$.
4. For all $x \in \mathbb{R}$, $-\mathbf{i} \arcsin(\mathbf{i}\, x) = \operatorname{argsinh}(x)$.

1.2 Parallel curves in the plane

The notion of a *parallel curve* to another is a natural generalization of the concept of parallel line. It was firstly defined in the work of Leibnitz at the end of the 17th century. In this section, a brief introduction to parallel curves—the definition and some properties—is given.

1.2.1 Definition of a parallel curve

Different definitions of parallel curves can be given (see e.g. [102], [98] or [52]). If α is the initial curve, its parallel curve at a distance d can be defined as the envelope of a family of congruent circles centered at the curve and of radius d (see Figure 1.1).

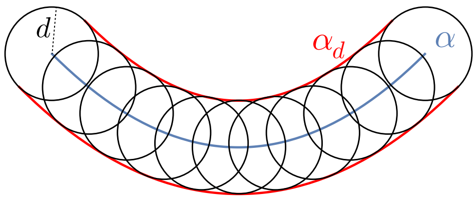

Figure 1.1: The parallel curve to α at a distance d as the envelope of the family of circles centered on α and of radius d. In the example shown in the figure, the parallel curve is made of two unconnected pieces.

Working with regular parameterized curves, another possible definition can be given—Definition 1.5—which is the one we will mainly use (see Figure 1.2).

Definition 1.5 (Parallel curve)**.** Let $\alpha : I \to \mathbb{R}^2$ be a positively oriented regular parametric curve and $d > 0$. The *(inner) parallel curve to α at a distance d* is defined as the curve $\alpha_d : I \to \mathbb{R}^2$ given by

$$\alpha_d(s) = \alpha(s) + d\,\mathbf{n}(s).$$

If the sign $+$ is replaced by $-$, it is the definition for the *outer parallel curve to α at a distance d.* If α is negatively oriented, these terms are interchanged.

Both given definitions of a parallel curve—as an envelope and as a parametric curve—are not entirely equivalent, [19]. The main difference is that the second one assumes differentiability in the initial curve. Moreover, as an envelope of circles, both parallels—the inner and the outer—are obtained, whereas as a constant normal distance from a simple parametric curve, only a connected component is described.

Parallel curves have been widely studied in the context of computer-aided geometric design, where the most common term to refer to this kind of curves

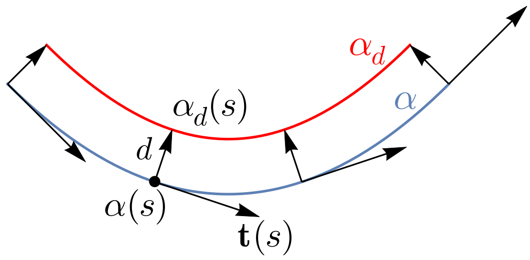

Figure 1.2: The inner parallel curve to α at a distance d as a parameterized curve.

is the name of *offset curve*. They are useful in many practical applications such as in numerically controlled machining or computer graphics.

Self-intersections—both, local and global—on offset curves are issues that must be controlled in some practical situations. If self-intersections cannot be avoided, then numerical methods to detect and trim them are also of interest (see [25] and Figure 1.3).

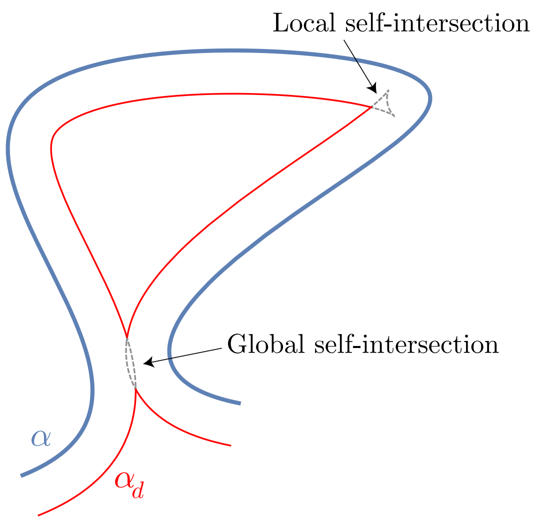

Figure 1.3: Offset curves may present self-intersections. There are algorithms to detect and trim these regions. Note that local self-intersections happen in big curvature regions for the initial curve.

Associated with the self-intersection problem it lies the study of singularities of offset curves (see [68], [30]).

Proposition 1.6. *Let $\alpha : I \to \mathbb{R}^2$ be a regular curve of curvature κ and let $d > 0$. If*

$$d < \frac{1}{\kappa_{max}},$$

where $\kappa_{max} = \sup_{t \in I} \left|\kappa(t)\right|$, then the (inner or outer) parallel curve to α at a distance d is regular.

Proof. Without loss of generality, assume α to be arc-length parameterized and positively oriented. We will deal with the case of inner and outer parallel curves at the same time:

$$\alpha_d(s) = \alpha(s) \pm d\,\mathbf{n}(s).$$

Since $\big\|\alpha_d'(s)\big\| = \big|1 \mp d\,\kappa(s)\big|$, we deduce that α_d is not regular if there exists $s_0 \in I$ such that

$$d = \frac{1}{\pm\kappa(s_0)}. \tag{1.1}$$

Of course, (1.1) can only be satisfied if $\pm\kappa(s_0) > 0$. Note that in regions of $\kappa > 0$, the outer parallel curves are regular (since $d \neq -1/\kappa < 0$) whereas in regions of $\kappa < 0$, the regular ones are the inner parallels (since $d \neq 1/\kappa < 0$). The fact is to avoid equality (1.1) when it can be achieved: $d = 1/\kappa(s_0)$ for $\kappa(s_0) > 0$ or $d = -1/\kappa(s_0)$ for $\kappa(s_0) < 0$. That can be done with the condition of the statement (notice the absolute value in the curvature, which avoids both possible cases). □

Remark 1.7. By the discussion given in the proof of Proposition 1.6, note that for convex curves, the outer parallels are always regular since $1+d\,\kappa(s) > 0$ for all $s \in I$.

From the computational geometry point of view, *Pythagorean-hodograph curves* are of interest in this setting, [29]. A polynomial (rational) curve α is called of *Pythagorean hodograph (PH)* if its hodograph, $\alpha'(t)$, is a polynomial (rational function) in t. Immediately, by its definition, any rational PH curve has rational parallel curves. Thus, PH curves allow an exact representation of their offsets which is easily implementable by computer. An expression for all rational curves with rational parallels was given in [71].

Consider now closed curves and their parallels. Two important geometric measures to ask for are the length and the area of a parallel curve in terms of the length and the area of the original one. These questions are answered in Theorem 1.8 (see Figure 1.4 to visualize it). Such results are due to J. Steiner, [88], and are usually known as *Steiner's formulae for parallel curves.* Steiner studied the volume for convex regions in the 2 and 3-dimensional Euclidean space and was especially interested in the case of having polygonal boundaries. As a more recent reference for the interested reader, see Chapter 1 and 10 of [34]. The statement below does not assume convexity (as Steiner did), but some assumptions are taken instead. Its proof uses elementary differential geometry.

Theorem 1.8 (Steiner's formulae for parallel curves)**.** *Let α be a positively oriented regular Jordan curve of curvature κ. Let α_d be the inner parallel curve to α at a distance $d > 0$. If α_d is regular and d is sufficiently small, then*

$$\mathcal{L}(\alpha_d) = \mathcal{L}(\alpha) - 2\,\pi\,d.$$

If, in addition, α_d is simple (it has no self-intersections), then

$$\mathcal{A}(\alpha_d) = \mathcal{A}(\alpha) - \mathcal{L}(\alpha)\, d + \pi\, d^2.$$

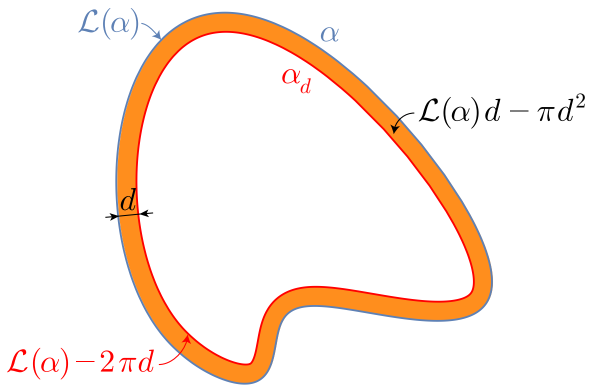

Figure 1.4: Representation of Steiner's formulae for parallel curves. The area and length of the parallel at a distance d is given in terms of those of the initial curve and d. The expressions given in the figure are for an inner parallel.

Proof. Without loss of generality, suppose that α is arc-length parameterized. We have that

$$\big\|\alpha_d'(s)\big\| = \big|1 - d\,\kappa(s)\big| \neq 0.$$

In addition, since α_d is regular, $1 - d\,\kappa(s)$ must have a constant sign. For $d = 0$, such a sign is positive, so for a sufficiently small d, $1 - d\,\kappa(s) > 0$ for all $s \in I$. Thus, the length of α_d is

$$\mathcal{L}(\alpha_d) = \int_I \big\|\alpha_d'(s)\big\|\ \mathrm{d}s = \int_I \big(1 - d\,\kappa(s)\big)\ \mathrm{d}s = \mathcal{L}(\alpha) - d \int_I \kappa(s)\ \mathrm{d}s.$$

Since α is closed and positively oriented,

$$\int_I \kappa(s)\ \mathrm{d}s = 2\,\pi \tag{1.2}$$

and the first result follows.

For the second statement, consider the region to compute the area as a parametric surface $\mathbf{x} : I \times [0, d] \to \mathbb{R}^2 \times \{0\}$ defined by

$$\mathbf{x}(s, u) = \alpha(s) + u\,\mathbf{n}(s).$$

Since

$$\mathbf{x}_s(s, u) = \big(1 - u\,\kappa(s)\big)\,\mathbf{t}(s), \quad \mathbf{x}_u(s, u) = \mathbf{n}(s),$$

we have that

$$(\mathbf{x}_s \wedge \mathbf{x}_u)(s, u) = \big(1 - u\,\kappa(s)\big)\,\mathbf{b}(s). \tag{1.3}$$

Note that (1.3) is not zero by hypothesis, so that $\mathbf{x}$ is regular. Therefore, since α_d is simple, the area of α minus the area of α_d is equal to

$$\iint_{I\times[0,d]} \big\|(\mathbf{x}_s \wedge \mathbf{x}_u)(s,u)\big\| \,\mathrm{d}(s,u) = \int_I \left(\int_0^d \big(1 - u\,\kappa(s)\big)\,\mathrm{d}u \right) \mathrm{d}s$$
$$= \int_I \left(d - \frac{d}{2}\,\kappa(s) \right) \mathrm{d}s = d\,\mathcal{L}(\alpha) - \frac{d^2}{2} \int_I \kappa(s)\,\mathrm{d}s.$$

By using (1.2) and arranging terms, the second result of the statement is found. □

Remark 1.9. The corresponding formulae of Theorem 1.8 for outer parallel curves are

$$\mathcal{L}(\alpha_d) = \mathcal{L}(\alpha) + 2\,\pi\,d \quad \text{and} \quad \mathcal{A}(\alpha_d) = \mathcal{A}(\alpha) + \mathcal{L}(\alpha)\,d + \pi\,d^2.$$

They are deduced just by taking the opposite normal direction in the definition of α_d; a sign that can be carried out by taking a distance $-d$ instead of d in the inner parallel formulae.

In the case of outer parallel curves to a convex curve, the formula

$$\mathcal{L}(\alpha_d) = \mathcal{L}(\alpha) + 2\,\pi\,d$$

holds for any length d since $1 + d\,\kappa(s) > 0$ for any $d > 0$.

There are extensions of Steiner's formulae for certain non-convex curves not considered in Theorem 1.8. The interested reader can see the work of Hadwinger, for instance [37] and [38].

1.2.2 Parallel curves according to an angle

An extension to the concept of parallel curves can be given by taking any angle instead of a right one (see Figure 1.5). See [14] as a reference where this kind of curves are considered.

Definition 1.10 (Parallel curve according to an angle)**.** Let $\alpha : I \to \mathbb{R}^2$ be a regular parametric curve, $d > 0$ and $\omega \in \,]-\pi, \pi]$. The *parallel curve to α at a distance d according to a constant angle ω* is defined as the curve $\alpha_{d,\omega} : I \to \mathbb{R}^2$ given by

$$\alpha_{d,\omega}(s) = \alpha(s) + d\,\big(\cos\omega\,\mathbf{t}(s) + \sin\omega\,\mathbf{n}(s)\big).$$

Readily, for a positively oriented curve α, its parallel curve at a distance d according to the angles $\pi/2$ and $-\pi/2$ correspond to the concepts of inner and outer (respectively) parallel curves.

The next theorem states that all parallel curves according to a non-right angle are regular. As far as the author knows, this result has not been stated before.

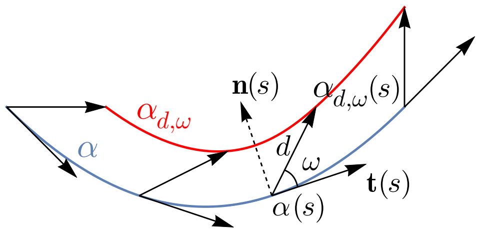

Figure 1.5: The parallel curve to α at a distance d according to a constant angle ω.

Theorem 1.11. *Let $\alpha : I \to \mathbb{R}^2$ be a regular parametric curve, $d > 0$ and $\omega \in \,]-\pi, \pi]$. Any parallel curve to α at a distance d according to the angle ω is always regular except for orthogonal inner parallels, where a sufficient condition for its regularity is to have*

$$d < \frac{1}{\kappa_{max}},$$

with $\kappa_{max} = \sup_{s\in I}\bigl|\kappa(s)\bigr|$.

Proof. Suppose, without loss of generality, that α is arc-length parameterized. Thus, since

$$\alpha'_{d,\omega}(s) = \bigl(1 - d\,\kappa(s)\sin\omega\bigr)\,\mathbf{t}(s) + d\,\kappa(s)\cos\omega\,\mathbf{n}(s),$$

it follows that

$$\begin{aligned}\bigl\|\alpha'_{d,\omega}(s)\bigr\|^2 &= \bigl(1 - d\,\kappa(s)\sin\omega\bigr)^2 + d^2\,\kappa^2(s)\cos^2\omega \\ &= 1 - 2\,d\,\kappa(s)\sin\omega + d^2\,\kappa^2(s) = \cos^2\omega + \bigl(\sin\omega - d\,\kappa(s)\bigr)^2.\end{aligned}$$

Therefore, if $\omega \neq \pm\pi/2$, then $\alpha_{d,\omega}$ is regular. The second part of the statement for orthogonal parallels ($\omega = \pm\pi/2$) is Proposition 1.6. □

Example 1.12. At first sight it can seem a little shocking the conclusion of Theorem 1.11. Let's illustrate it with an example. Consider a piece of a parabola as initial curve: $\alpha(t) = \bigl(t, t^2\bigr)$, $t \in [-3/2, 3/2]$. The parallel curve to α at a distance d according to an angle ω can be easily computed:

$$\alpha_{d,\omega}(t) = \left(t + d\,\frac{\cos\omega - 2\,t\sin\omega}{\sqrt{1+4\,t^2}},\, t^2 + d\,\frac{2\,t\cos\omega + \sin\omega}{\sqrt{1+4\,t^2}}\right).$$

As shown in Theorem 1.11, singularities can only appear in $\alpha_{d,\omega}$ if $\omega = \frac{\pi}{2}$. In Figure 1.6 the curve $\alpha_{d,\omega}$ is plotted for a distance

$$d = 1 > \frac{1}{\kappa_{\max}},$$

where $\kappa_{\max} = 2$ is the maximum curvature of α.

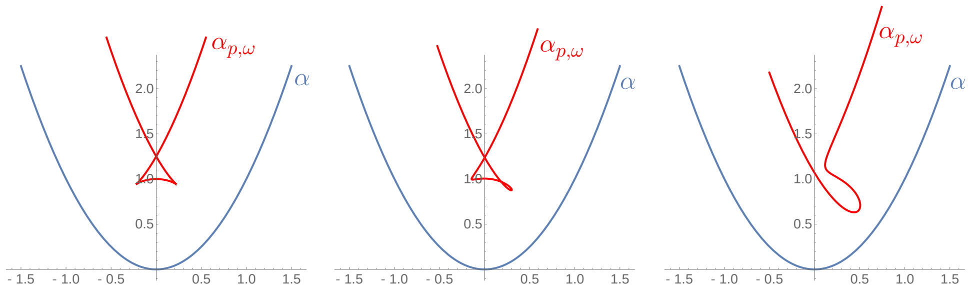

Figure 1.6: Some parallel curves $\alpha_{d,\omega}$ to a parabola $\alpha(t) = (t, t^2)$ at a distance $d = 1$ according to constant angles $\omega = \frac{\pi}{2}, \frac{15\,\pi}{32}$ and $\frac{3\,\pi}{8}$, respectively. Only for a right angle singularities can appear. For another angle, regularity is ensured but self-intersections may happen.

Similarly as in Theorem 1.8, a formula for the area of a positively oriented parallel curve according to any constant angle can be deduced:

$$\mathcal{A}(\alpha_d) = \mathcal{A}(\alpha) - \mathcal{L}(\alpha)\, d \sin\omega + \pi\, d^2.$$

In Section 3.5, a general setting will be considered and this result will be deduced as a particular case—Theorem 3.52.

1.3 Constant width curves in the plane

In this section, a special kind of curves with many applications are defined. As main references, the reader can see Chap. 25 of [74], Chap. 7 of [101], Sec. 11 of [53] and Chap. 1 of [85].

1.3.1 The width of a curve

An arbitrary shape has different widths depending in which direction the measure is done. Thus, to give a definition of width for planar curves, a direction must be set.

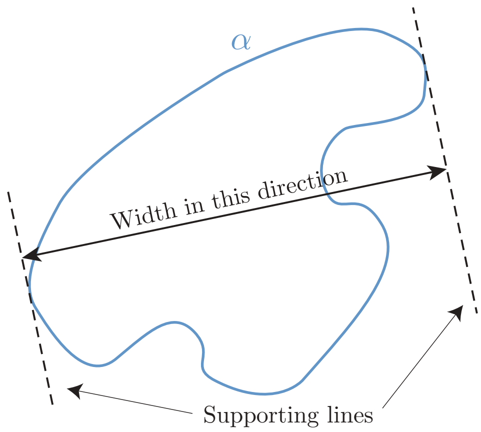

Figure 1.7: Definition of the width of a curve as the perpendicular distance between two supporting lines.

Figure 1.7 illustrates the definition of the *width* of a curve measured in some direction as the perpendicular distance between two *supporting lines* which “contains” the curve in between just “touching” it. The formal definition, [74], is given below.

Definition 1.13 (Supporting line of a curve in a direction)**.** Given a closed curve α and a direction $\mathbf{v}$, a line is called a *supporting line of* α *in the direction* $\mathbf{v}$ if it is orthogonal to the direction $\mathbf{v}$, it has at least one point in common with the curve α and the entire curve lies on only one side of such a line.

A supporting line is not the same as a tangent line. Tangent lines are not always supporting lines (see Figure 1.8-left) and the converse is false either (see Figure 1.8-right).

A closed curve has exactly two supporting lines in each direction, [74]. Indeed, given some direction, the idea is to project each point of the curve orthogonally on a line parallel to that direction. This projection will be a segment whose endpoints determine each supporting line. See Figure 1.9 for a visualization. Thus, it makes sense the following definition.

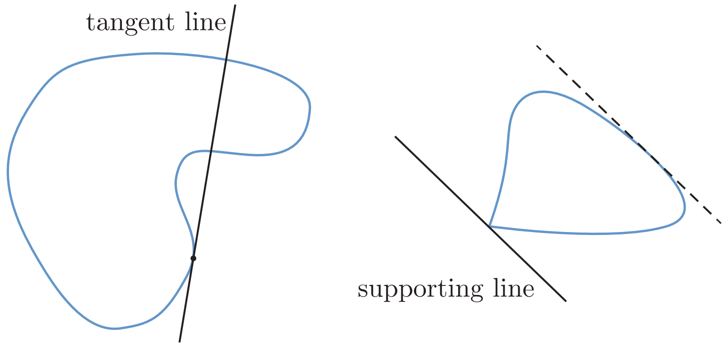

Figure 1.8: On the left, a tangent line which is not a supporting line. On the right, a supporting line which is not a tangent.

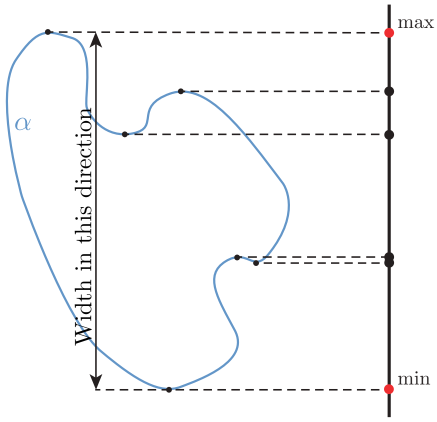

Figure 1.9: Given a direction to measure the width of a curve, the supporting lines are determined by the maximum and minimum value for the height of the orthogonal projections for any point of the curve. Such a width is the difference between these two values. In the figure, only the tangent orthogonal projections have been represented, but the projection is done for every point of the curve α.

Definition 1.14 (Width of a curve in a direction)**.** The *width of a closed curve in some direction* is defined as the distance between the supporting lines in such direction.

We are interested in those shapes with a constant width regardless the direction in which that width is measured. These are called constant width curves. Some authors restrict the definition to convex curves, [101]. That is the definition below.

Definition 1.15 (Curve of constant width)**.** A closed convex curve is called a *curve of constant width (CCW)* if its width is the same for all directions. That constant value is called the *width of the curve*.

Remark 1.16. If α is a regular curve, the supporting lines of α in a direction are parallel tangent lines to α (see Figure 1.10-left). Nevertheless, in convex geometry it is more appropriate to define the width of a curve with supporting lines (not with tangent lines) because, in that way, smoothness on the curve is not needed. Thus, famous shapes as the *Reuleaux triangles* (see Figure 1.10-right), which are not regular curves, are well defined as constant width curves.

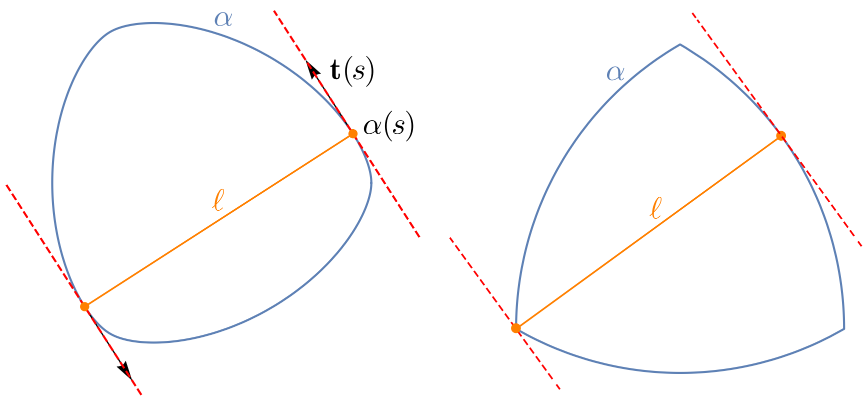

Figure 1.10: Two curves of constant width ℓ. The supporting lines are tangent lines to the curve in the smooth case (left) whereas in the Reuleaux triangle (right) their tangent lines are not well defined for the corners (unlike the supporting lines).

In the examples of Figure 1.10, the chord joining the contact points of both supporting lines and which measures the width is always perpendicular to these lines. That is not by chance (a proof from [101] is reproduced below).

Proposition 1.17. *In a curve of constant width, each chord joining the contact points of the parallel supporting lines is perpendicular to those lines and hence has length the constant width of the curve.*

Proof. Let α be a constant width curve. Let l_1 and l_2 be two parallel supporting lines of α with points A_1 and A_2, respectively, in common with α (see Figure 1.11). If the chord A_1A_2 were not perpendicular to the supporting lines, then the distance d between these lines (the width in that direction) would be less than the length of A_1A_2 and, therefore, than the distance d' between the supporting lines l_1' and l_2' of α which are perpendicular to A_1A_2 (d' is another width). That is a contradiction because α is of constant width. $\square$

Remark 1.18. Thanks to Proposition 1.17, equivalent definitions for constant width curves can be given. For instance: *a positively oriented closed curve is called of constant width d if it coincides with its inner parallel curve at a distance d.*

If α is a regular closed convex curve and P and Q are two distinct points on α such that the tangents at P and Q are parallel, then the segment PQ is called by some authors *a diameter of α*. If the diameter is constant for all such pairs P and Q, then α is called a *curve of constant diameter* (see

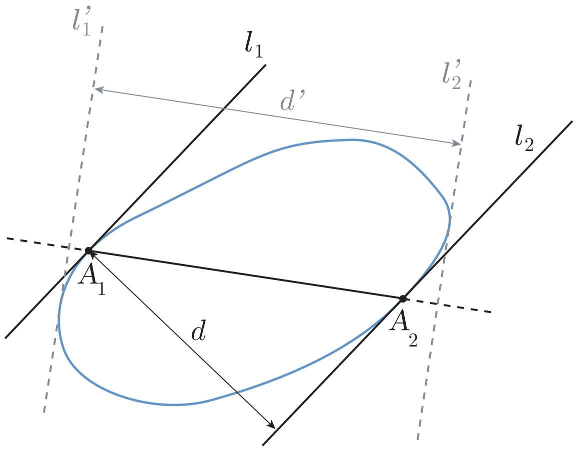

Figure 1.11: Proof of Proposition 1.17. The only way the statement not to be true is having a curve of variable width.

for instance [12] or [62]). Readily, any curve of constant diameter is a curve of constant width. But by Proposition 1.17 the converse is also true: any regular curve of constant width is a curve of constant diameter. Thus, both definitions are the same.

1.3.2 Support function for convex curves

It can be difficult to work and study constant width curves if a proper parameterization for them is not chosen. In this section, a parameterization for convex curves by means of what is called a *support function* is described. Let $\alpha : [0, 2\pi] \to \mathbb{R}^2$ be a closed convex curve (extended its domain to $\mathbb{R}$ by periodicity). Let's parameterize α by a support function $h(t) > 0$ of class $\mathcal{C}^1$. The usual procedure to do that consists of computing the curve α as the envelope of the family of supporting lines (see [41] or [73]). Next, a direct parameterization without that envelope is described.

First, we choose a point inside the curve, suppose that this point is the origin O. Given $t \in [0, 2\pi]$, consider the line generated by the direction $\mathbf{v}(t) = \big(\cos(t), \sin(t)\big)$ from O and consider the supporting lines of α in that direction. Call P the intersection point between one of these supporting lines and the line generated by $\mathbf{v}(t)$. Denote by $\alpha(t)$ the point in common of the selected supporting line with α. If $h(t)$ is the distance between O and P and we denote by $p(t)$ the distance between P and $\alpha(t)$, then α can be parameterized by

$$\alpha(t) = h(t)\,\big(\cos(t), \sin(t)\big) + p(t)\,\big(-\sin(t), \cos(t)\big).$$

See in Figure 1.12 the described procedure to parameterize α. We know that for regular curves the supporting lines coincide with the tangent lines. Therefore, for regular curves, $\alpha'(t)$ must be perpendicular to $\mathbf{v}(t)$. Thus,

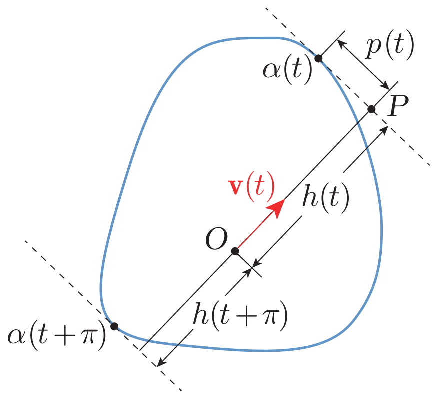

Figure 1.12: Parameterization of α by means of a support function.

since

$$\alpha'(t) = \big(h'(t) - p(t)\big)\,\big(\cos(t), \sin(t)\big) + \big(h(t) + p'(t)\big)\,\big(-\sin(t), \cos(t)\big),$$

we deduce that $p(t) = h'(t)$. With that, the parameterization by a support function h of a regular convex curve takes the form

$$\alpha(t) = h(t)\,\big(\cos(t), \sin(t)\big) + h'(t)\,\big(-\sin(t), \cos(t)\big), \quad t \in [0, 2\pi]. \tag{1.4}$$

Of course, with this setting, $\alpha(t+\pi)$ is the other point in common with the other supporting line and $h(t+\pi)$ its distance to O. Thus, $h(t) + h(t+\pi)$ measures the width of the curve α.

A choice of a support function h such that $h(t)+h(t+\pi)$ is constant will lead to a constant width curve parameterization. For instance, a support function of the kind

$$h(t) = \frac{a}{2}\cos^2\left(\frac{3\,t}{2}\right) + b = \frac{a}{2}\,\cos(3\,t) + \frac{a}{2} + b,$$

verifies

$$h(t) + h(t+\pi) = a + 2\,b.$$

The example of Figure 1.10-left is constructed with this support function for $a = 1$ and $b = 5$.

Theorem 1.19 (Barbier)**.** *If α is a curve of constant width ℓ, then*

$$\mathcal{L}(\alpha) = \pi\,\ell.$$

The theorem above is due to Barbier, [4]. There are different proofs for this result (the interested reader can see, for example, the approach of [101] or [53] or the one of [42]). A simple proof can be given if α is a regular curve parameterized by a $\mathcal{C}^2$ support function h, [85]. In such a case, it is easy to see that

$$\big\|\alpha'(t)\big\| = h(t) + h''(t) > 0.$$

Hence,

$$\begin{aligned}\mathcal{L}(\alpha) &= \int_0^{2\pi} h(t)\ \mathrm{d}t = \int_0^{\pi} h(t)\ \mathrm{d}t + \int_\pi^{2\pi} h(t)\ \mathrm{d}t \\ &= \int_0^{\pi} h(t)\ \mathrm{d}t + \int_0^{\pi} h(t+\pi)\ \mathrm{d}t = \int_0^{\pi} \big(h(t) + h(t+\pi)\big)\ \mathrm{d}t = \pi\,\ell.\end{aligned}$$

In Section 3.5, another proof of Barbier's theorem will be given as a particular case of a more general setting without using a support function.

1.4 Bicycle curves in the plane

The curves generated by bicycle tire tracks have been widely studied in the last decades, see for example [31] and [22]. The bicycle is usually represented by a moving oriented segment in such a way that its wheels are represented by the endpoints. Moreover, it is required such segment to be always tangent to the path of the rear wheel curve (see Figure 1.13).The length ℓ of a bicycle will be referred to as the fixed distance between both wheels (the wheelbase).

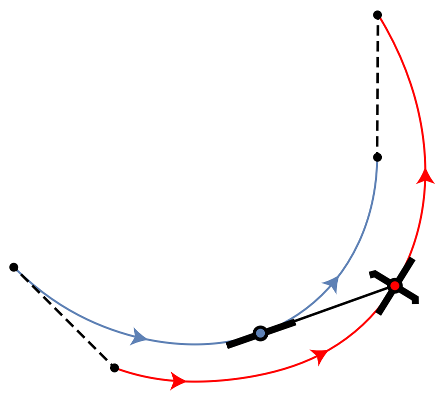

Figure 1.13: The bicycle is always in the tangent direction of the rear tire track.

Thus, if α_R and α_F represent the rear and the front wheel, respectively, of a bicycle of length ℓ, then

$$\alpha_F(t) = \alpha_R(t) + \ell\, \mathbf{t}_R(t),$$

where $\mathbf{t}_R$ represents the tangent vector of the rear track α_R. Note, by the definition, that the front track α_F is defined as a parallel curve to α_R at a distance ℓ according to a constant angle $\omega = 0$. Therefore, if the rear track is given, the study of the front track can be seen as a particular case of the curves defined in Section 1.2.2.

Sometimes, the rear and front tracks are described in such a way that one cannot know in which direction the bicycle went. In such a case, the pair of tire tracks are called ambiguous. More precisely, a pair of curves α and γ is called *ambiguous* if they can serve the trajectories of the rear and front bicycle wheels when traversed into two opposite directions (see [90]). The case of ambiguous curves is closely related to a kind of Holditch curves (those which coincide with the envelope of all the chords of constant length). The notion of Holditch curve is introduced in the next chapters.

Chapter 2

Historial review on Holditch's theorem

Holditch's theorem is a geometrical result on areas of planar curves published in 1858. The intriguing conclusion of the theorem was in the spotlight of many mathematicians in that time and it was widely studied among variations and generalizations. Nevertheless, the theorem was forgotten and a lot of research done in the field ceased to be so well known today. The objective of this chapter is to compile some of this old research related to Holditch's theorem and to present those results in a clear display and modern language. In some cases, the results are stated in a rough way, leaving the formalism on the definitions or the statements to be given in a further work or in the next chapters.

2.1 Introduction

Rev. Hamnet Holditch was an English mathematician born in 1800 at Lynn (Norfolk, England). He studied mathematics at the Gonville and Caius College (University of Cambridge) and obtained his bachelor's degree (B.A.) in 1822 with very good results—in fact, he was the Senior Wrangler and awarded with the Smith's prize of his year. He earned his master's degree (M.A.) in 1825.

Ten years later, in 1835, Holditch became the president of the same college in which he studied and stayed on that position until his death on December 12, 1867.

He lived in a scientific and industrial expansion time, so motivated by the rods, wheels and mechanisms being used for invention, the mathematical properties of curves and surfaces became very interesting for him. Probably due to this reason, most of Holditch's research was centered in the field of geometry. His most famous result is what he called *Geometrical theorem*, [39], today known as Holditch's theorem, a result on areas in plane geometry.

If a chord of fixed length is allowed to rotate until a full turn with its extremities on a convex closed curve, Holditch's theorem states that the locus of a tracing point on the chord a distance p from one end and a distance q from the other is a closed curve whose area is less than that of the original curve by $\pi\, p\, q$. That area is sometimes referred to as the *Holditch area* and the curve generated by the tracing point as the *Holditch curve*.

Holditch's theorem was included as one of the 250 milestones in the history of Mathematics in [69]. In that book, the author quotes the question written by Mark J. Cooker in [13]:

> *Two things immediately struck me as astonishing. First, the formula for the area is independent of the size of the given curve. Secondly, $\pi\, p\, q$ is the area of an ellipse of semi-axes p and q, but there are no ellipses in the theorem!*

At first sight, making a quick search in online databases, it could seem that there are not so many references to Holditch's theorem or some research done in that field. It is true that recent articles such as [72] can be found. Nevertheless, if a deeper search is done, many articles related to Holditch's theorem can be discovered, especially in the years close to Holditch, when his result became very popular.

The aim of this chapter is to review Holditch's theorem and some of its variations and generalizations given along the old years making them more accessible and presented together in a clear way.

2.2 First statements of Holditch's theorem

The first appearance of Holditch's theorem in the literature was in *The Lady's and Gentleman's Diary* in the year 1857 as the XV prize problem for resolution proposed by someone under the name of Petrarch (maybe a pseudonym of Holditch?). The original problem, found in [99], page 72, is stated as follows:

> *A rod CC' of a given length has its two ends in the curve of an ellipse and moved round, having a tracing point P, at the distances c and c' from its ends, tracing a curve. Show that the area contained between the curve and the ellipse $= \pi\, c\, c'$, and is therefore independent of the ellipse.*

As it can be seen, the original proposed problem is restricted to the case of the initial curve being an ellipse. Some answers for the problem were presented in the next volume, [100], of the same journal in the year 1858 (pages 65–69) and, in fact, in some of them a more general result was proven. More than six solutions for the problem were received, but only five were published in the journal due to lack of space. Some of the solutions gave the result that today is known as the *classical Holditch theorem*. Other solutions were even a generalization of this result. In addition, one of the general solutions given in that journal was due to its editor, Mr. Woolhouse, in the page 96 (with date 21st of November of 1856, so it was the first one). For that, the first extension of the classical Holditch theorem is sometimes known as *Woolhouse's theorem*.

At the same time, Holditch published its proof in *The Quarterly Journal of Pure and Applied Mathematics*, [39], page 38. The result presented there is the *classical Holditch theorem*, which was stated as follows:

> *If a chord of a closed curve of constant length $c + c'$, be divided into two parts of lengths c, c' respectively, the difference between the areas of the closed curve, and of the locus of the dividing point, will be $\pi\, c\, c'$.*

As said by Broman in [11], there were some implicit and important assumptions in the original proof which should have been considered, as it can be found in the cited article and in [10]. More recently, in [72], this discussion is also addressed.

During some years after its publication, Holditch's theorem was very known by many mathematicians and it appeared normally in some calculus textbooks. Furthermore, about twenty years later from the original publication, there were some authors interested in kinematics who gave some related results, as for example the extensions given by Leudesdorf or by Elliott. Some of these new developments also appeared in books such as Williamson's integral calculus [97] of 1880 or, later, in Edwards' treatise on integral calculus [24] of 1921.

2.2.1 The classical Holditch theorem

The name of the classical Holditch theorem refers to the result given in [39] by Holditch. The proof that will be given here follows the same idea of the extension presented in Section 2.2.2 but in a particular case. More proofs are available of the classical result (as the one given by Holditch) based on the usage of polar coordinates. That point of view will be discussed in Section 2.2.3.

The revolutions of the moving chord will be assumed to be done counterclockwise. In current language, the theorem can be stated as follows (see in Figure 2.1 its representation).

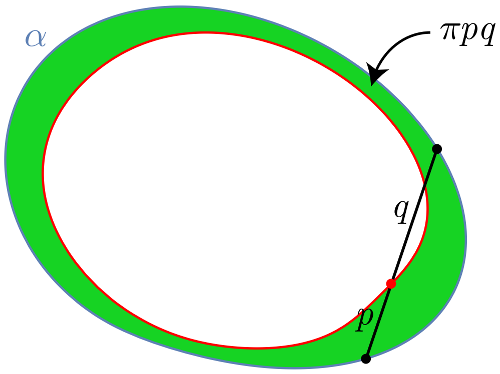

Figure 2.1: Representation of Holditch's theorem.

Theorem 2.1 (Classical Holditch's theorem)**.** *If a chord of fixed length moves smoothly with its extremities on a simple closed regular curve α in a complete (counterclockwise) revolution, having a tracing point P at the distances p and q from its ends, then the difference between the areas of α and of the locus of the dividing point is $\pi\, p\, q$.*

Proof. Let $\alpha : I \to \mathbb{R}^2$, $\alpha(s) = \big(x(s),\, y(s)\big)$, be a simple convex closed regular curve parameterized by arc length. Consider a chord of fixed length $\ell = p+q$ that slides smoothly a full positive turn with both endpoints on α. Denote by γ the curve traced out by the tracing point P, the Holditch curve. Notice that α and γ must be positively oriented. The Holditch curve γ can be described as

$$\gamma(s) = \alpha(s) + p\,\big(\cos\theta(s),\, \sin\theta(s)\big), \quad s \in I,$$

where $\theta(s)$ is the oriented angle function from the positive OX axis to the moving chord placed at $\alpha(s)$. Let's compute the area enclosed by γ:

$$\mathcal{A}(\gamma) = \int_I \big(x(s) + p\cos\theta(s)\big)\big(y'(s) + p\,\theta'(s)\cos\theta(s)\big)\,\mathrm{d}s.$$

Expanding this expression, we get

$$\begin{aligned}\mathcal{A}(\gamma) = \mathcal{A}(\alpha) + p \int_I \big(\theta'(s)\, x(s) + y'(s)\big) \cos\theta(s)\ \mathrm{d}s \\ + p^2 \int_I \theta'(s) \cos^2\theta(s)\ \mathrm{d}s. \qquad (2.1)\end{aligned}$$

Since

$$\int_I \theta'(s)\cos^2\theta(s)\ \mathrm{d}s = \pi,$$

the previous expression can be rewritten as

$$\mathcal{A}(\gamma) = \mathcal{A}(\alpha) + p \int_I \big(\theta'(s)\, x(s) + y'(s)\big) \cos\theta(s)\ \mathrm{d}s + \pi\, p^2. \qquad (2.2)$$

Now, notice that

$$\alpha(s) + \ell\,\big(\cos\theta(s),\, \sin\theta(s)\big)$$

is another parameterization of the original curve. Thus, similarly,

$$0 = \ell \int_I \big(x(s)\,\theta'(s) + y'(s)\big)\cos\theta(s)\ \mathrm{d}s + \pi\,\ell^2,$$

that is to say,

$$\int_I \big(x(s)\,\theta'(s) + y'(s)\big)\cos\theta(s)\ \mathrm{d}s = -\pi\,\ell.$$

Therefore, substituting this on (2.2) and arranging terms, the desired result is obtained:

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = \pi\, p\,(\ell - p).$$

□

2.2.2 Woolhouse's extension of Holditch's theorem

As said in the beginning of Section 2.2, the first extension of Holditch's theorem was given by Woolhouse, [100], and later the result would be included in Williamson's calculus of 1880, [97]. The result was forgotten and Broman gave the same generalization in 1981, [11], but doing first in [10] a discussion about the existence of the angle parameter in the theorem. Without no reference to the previous authors, Cooker gave in 1998 the same general result in his paper [13]. The proofs given by Woolhouse, Williamson, Broman and Cooker are essentially the same and it is the one which will be reproduced here.

This extension of Holditch's theorem can be stated as follows. Notice the usage of *signed areas* (see Section 1.1.1).

Theorem 2.2 (Woolhouse)**.** *If a chord of fixed length moves smoothly with its extremities on two fixed closed curves α and $\bar{\alpha}$ in a complete (counter-clockwise) revolution, having a tracing point P at the distances p and q from its ends, then the signed areas given by the curves α, $\bar{\alpha}$ and the one generated by P, γ, satisfy the relation*

$$\tilde{\mathcal{A}}(\gamma) = \frac{q\,\tilde{\mathcal{A}}(\alpha) + p\,\tilde{\mathcal{A}}(\bar{\alpha})}{p+q} - \pi\,p\,q.$$

Proof. Let $\alpha : I \to \mathbb{R}^2$, $\alpha(s) = \big(x(s), y(s)\big)$, be one of the initial curves and let $\theta(s)$ be the function that for each $s \in I$, gives the positively oriented angle from the positive OX axis to the moving chord at $\alpha(s)$. The Holditch curve can be written as

$$\gamma(s) = \alpha(s) + p\,\big(\cos\theta(s),\, \sin\theta(s)\big), \quad s \in I.$$

The area enclosed by γ is

$$\tilde{\mathcal{A}}(\gamma) = \int_I \big(x(s) + p\cos\theta(s)\big)\big(y'(s) + p\,\theta'(s)\cos\theta(s)\big)\,\mathrm{d}s.$$

As in the classical statement proof, expanding this expression, we get

$$\tilde{\mathcal{A}}(\gamma) = \tilde{\mathcal{A}}(\alpha) + p\int_I \big(\theta'(s)\,x(s) + y'(s)\big)\cos\theta(s)\,\mathrm{d}s + \pi\,p^2. \tag{2.3}$$

If $\ell = p + q$, we also have that

$$\bar{\alpha}(s) = \alpha(s) + \ell\,\big(\cos\theta(s),\, \sin\theta(s)\big),$$

so that, analogously,

$$\tilde{\mathcal{A}}(\bar{\alpha}) = \tilde{\mathcal{A}}(\alpha) + \ell\int_I \big(\theta'(s)\,x(s) + y'(s)\big)\cos\theta(s)\,\mathrm{d}s + \pi\,\ell^2. \tag{2.4}$$

If we subtract (2.4) multiplied by p to (2.3) multiplied by ℓ, we get

$$\ell\,\tilde{\mathcal{A}}(\gamma) - p\,\tilde{\mathcal{A}}(\bar{\alpha}) = (\ell - p)\,\tilde{\mathcal{A}}(\alpha) - \pi\,\ell\,p\,(\ell - p).$$

Arranging this last equation the desired result is deduced. □

Theorem 2.2 is, indeed, a generalization of Holditch's theorem because if $\alpha = \bar{\alpha}$, then the classical statement (Theorem 2.1) is deduced:

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = \pi\,p\,q.$$

Note that for a single curve α, the counterclockwise movement of the chord forces the curves α and γ to be positively oriented, so that their signed areas are positive.

Theorem 2.2 can be easily extended to the case of a moving chord making n chord revolutions, as shown in [97] (n is the number of counterclockwise full revolutions minus the number of clockwise full revolutions). In this case,

$$\int_I \theta'(s) \cos^2 \theta(s)\, \mathrm{d}s = \pi\, n$$

so that the conclusion of the statement is

$$\tilde{\mathcal{A}}(\gamma) = \frac{q\,\tilde{\mathcal{A}}(\alpha) + p\,\tilde{\mathcal{A}}(\bar{\alpha})}{p+q} - n\,\pi\,p\,q. \tag{2.5}$$

Notice that this general scenario allows the chord not to make even a full revolution. See for example the situation of Figure 2.2. In such a case, we have $n = 0$, $\tilde{\mathcal{A}}(\alpha) = 0$ and $\tilde{\mathcal{A}}(\bar{\alpha}) = \pi$.

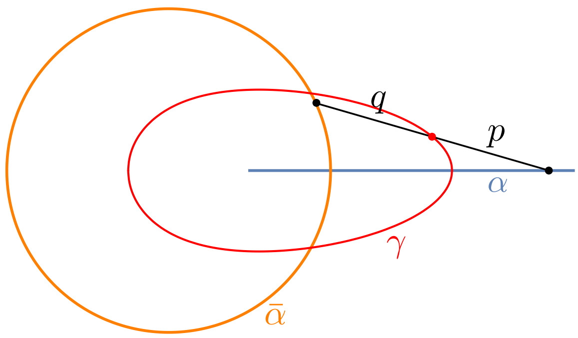

Figure 2.2: Example of a circle and a double-traced segment as initial curves for a chord length $\ell = p + q$ with $p = \ell/2$. Notice that the segment is a degenerated case of a closed curve.

Of course, in the case $\alpha = \bar{\alpha}$, Equation (2.5) turns into

$$\tilde{\mathcal{A}}(\alpha) - \tilde{\mathcal{A}}(\gamma) = n\,\pi\,p\,q,$$

which is the classical Holditch theorem for any number of chord revolutions n.

2.2.3 Use of polar coordinates in Holditch's theorem

Some of the other solutions published to the original problem and even the proof given by Holditch used the representation of curves with polar coordinates. Actually, in the original Holditch's proof, non-trivial properties about areas were used without any comment. The main fact is the usage of polar coordinates with a moving center to compute areas. That procedure is indeed correct and it can be justified—see for instance Mamikon's method of sweeping tangents, [1]. Next, the proof of Theorem 2.1 is reproduced following this approach, which also considers the envelope generated by the family of all the positions for the moving chord (see in Figure 2.3 an illustration for this proof).

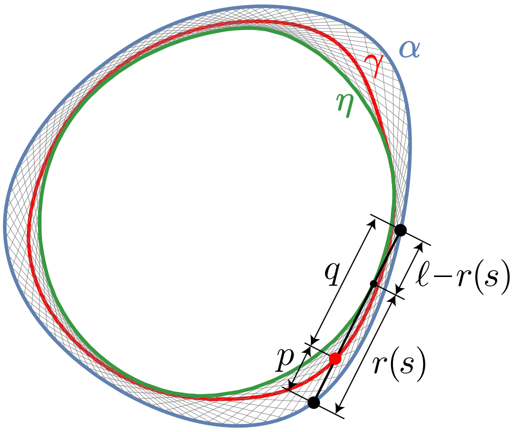

Figure 2.3: The initial closed curve α, the Holditch curve γ and the envelope η of all the chords of constant length $\ell = p + q$.

Proof of Theorem 2.1 with polar coordinates. Recall that the area enclosed by a closed curve given by its polar coordinates $r(\theta)$ can be computed with

$$\frac{1}{2}\int_0^{2\pi} r^2(\theta)\,\mathrm{d}\theta. \tag{2.6}$$

Let α be the initial curve. Consider the envelope η of all the segments of fixed length placed with their extremities in α, i.e. the curve which is tangent to all these directions. Mamikon's method of sweeping tangents allows us to work with polar coordinates with a moving center (the envelope) to compute areas according to an expression of the kind (2.6). If $r(s)$ is the function that gives the distance from the envelope $\eta(s)$ to the point $\alpha(s)$, then we have

$$\mathcal{A}(\alpha) - \mathcal{A}(\eta) = \frac{1}{2}\int_0^{2\pi} r^2(s)\,\mathrm{d}s, \tag{2.7}$$

and

$$\mathcal{A}(\alpha) - \mathcal{A}(\eta) = \frac{1}{2}\int_0^{2\pi} \big(\ell - r(s)\big)^2\,\mathrm{d}s.$$

With these two equations, it is found that

$$\int_0^{2\pi} r(s)\,\mathrm{d}s = \pi\,\ell. \tag{2.8}$$

Also, if γ is the Holditch curve, then

$$\mathcal{A}(\gamma) - \mathcal{A}(\eta) = \frac{1}{2}\int_0^{2\pi} \big(r(s) - p\big)^2\,\mathrm{d}s. \tag{2.9}$$

Now, with (2.7) and (2.9) and using (2.8) it is concluded that

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = \pi\,p\,(\ell - p).$$

□

2.3 Holditch's theorem and kinematics

Mr. Leudesdorf studied a problem in kinematics, [57], and the result he found was really a generalization of Woolhouse's theorem. This extension and many others related to kinematics are compiled on Chapter XV of Edwards' book [24] of 1921, where the interested reader can take a look. Here, Leudesdorf's result of the paper [57] will be presented but with the proof he gave on [58], which is simpler. Before that, the definition of the power of a point to a circle is needed for Leudesdorf's statement. Later, useful properties are regarded (see e.g. [89] as a reference).

2.3.1 Power of a point to a circle and some properties

Given two points P and Q in the plane, denote by $\overrightarrow{PQ}$ the vector from P to Q and by PQ the length of such a vector, i.e. $PQ = \|\overrightarrow{PQ}\|$.

Definition 2.3 (Power of a point to a circle)**.** The *power of point P to a circle* centered at a point O and of radius r is defined by

$$\mathcal{T} := OP^2 - r^2.$$

Remark 2.4. Note that from the definition, if the point P lies on the circle, it has zero power. If P lies outside the circle, it has positive power, but if it lies inside, the power is negative.

If the point P lies outside the circle, then the power of P to the circle has a simple geometrical interpretation: the squared radius of the circle centered at P that intersects orthogonally the first circle (see Figure 2.4). Since it is the distance t in the tangent line from P to the circle, but squared, sometimes it is called *the power of the tangent from P to the circle* and thus it can be written as $\mathcal{T} = t^2$.

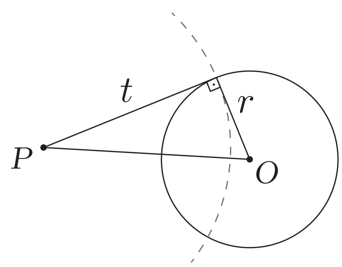

Figure 2.4: If the point P is outside the circle, the power of P to the circle is $\mathcal{T} = t^2$ by the Pythagorean theorem.

If the point P lies inside the circle, it seems that there is no particularly good geometric interpretation.

This definition is enough to state Leudesdorf's theorem of Section 2.3.2. Nevertheless, two important and useful results are worth to be given before that. The main fact is to give an alternative expression for the power of a point to a circle (Theorem 2.6).

The next theorem is known as the *intersecting secants theorem* if the point P lies outside the circle or *intersecting chords theorem* otherwise. To prove it, a well-known property of angles in a circle is needed (namely, a consequence of the *inscribed angle theorem*, [63], Figure 2.5): all the angles inscribed in a circle subtending the same arc are equal.

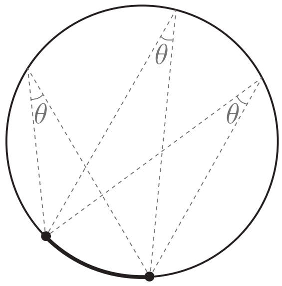

Figure 2.5: All the angles inscribed in a circle subtending the same arc are equal.

Theorem 2.5. *Let P be a point on the plane and consider a circle. Take two lines l and l' passing through P and cutting the circle. Let U and V be the two intersection points of l with the circle and U' and V' the same for l'. Then*

$$PU\,PV = PU'\,PV'.$$

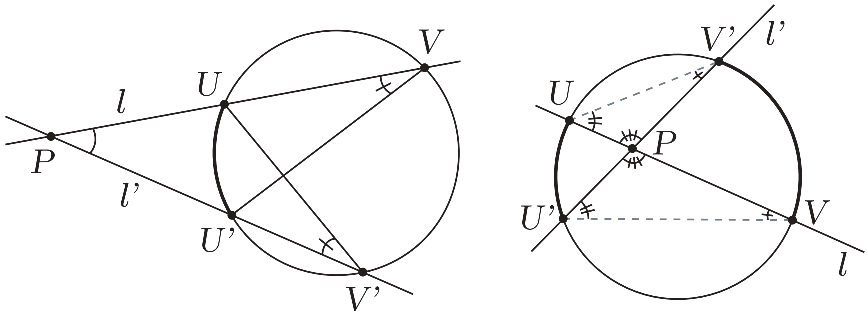

Figure 2.6: Setting of Theorem 2.5 depending if the point P is outside the circle (left) or inside it (right).

Proof. Suppose first that the point P is outside the circle (intersecting secants theorem, see Figure 2.6-left). The triangles PUV' and $PU'V$ are similar (because they have two equal angles; apply the consequence of the inscribed angle theorem to the arc UU'). Therefore,

$$\frac{PU}{PU'} = \frac{PV'}{PV},$$

which can be rewritten as in the statement.

Suppose now that the point P is inside the circle (intersecting chords theorem, see Figure 2.6-right). Again, by the consequence of the inscribed angle theorem (to the arcs UU' and VV'), we have that the triangles $PU'V$ and $PV'U$ are similar (because they have two equal angles). Then

$$\frac{PU}{PV'} = \frac{PU'}{PV}$$

so the statement is deduced. □

The power of a point has a really beautiful property: it can be obtained by multiplying the two lengths defined by the intersection of the circle and an arbitrary line passing through P with a positive sign if P lies outside the circle or with a negative sign otherwise (the result does not depend on the considered line thanks to Theorem 2.5). That property is known as the *power of a point theorem*, due to Jacob Steiner, [89].

Theorem 2.6 (Power of a point theorem)**.** *Let P be a point on the plane and consider a circle. A line passing through P cuts the circle at two distinct points, U and V. The power of the point P to the circle is*

$$\mathcal{T} = \pm PU\,PV$$

with a sign $+$ if P lies outside the circle or a sign $-$ otherwise.

Proof. By Theorem 2.5, it suffices to prove the result for the line that passes through P and the center O of the circle. In that case, if P is outside the circle (see Figure 2.7-left), then

$$PU\,PV = (PO - r)\,(PO + r) = PO^2 - r^2 = \mathcal{T}.$$

If P is inside the circle (see Figure 2.7-right), then

$$PU\,PV = (r - PO)\,(r + PO) = r^2 - PO^2 = -\mathcal{T}.$$

□

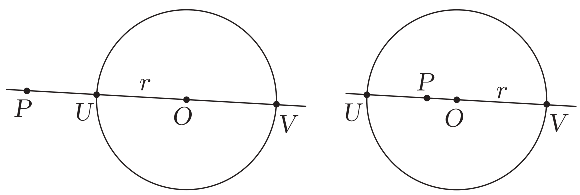

Figure 2.7: Setting of Theorem 2.6 depending if the point P is outside the circle (left) or inside it (right).

2.3.2 Theorems on kinematics

Consider the motion of a lamina over another fixed one making n full revolutions. Some chosen points on the moving lamina describe trajectories on the fixed one as it slides over it. A relation on the (signed) areas of the generated trajectories is given in Leudesdorf's theorem (see in Figure 2.8 a visualization).

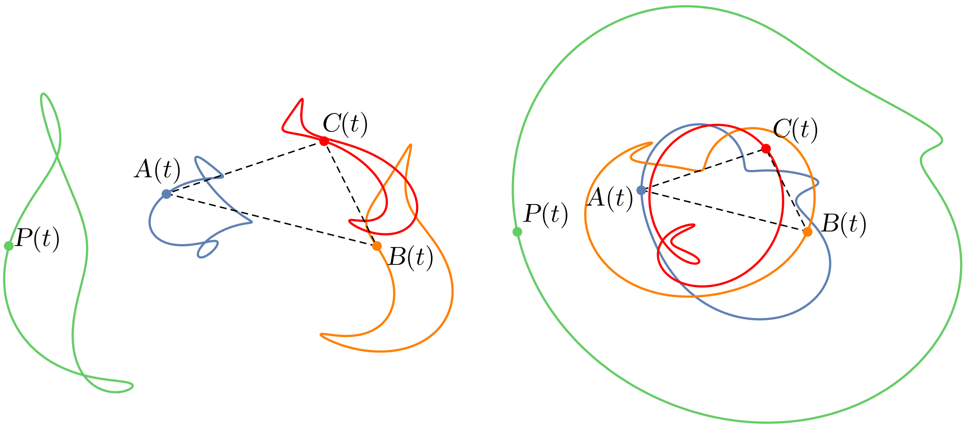

Figure 2.8: The moving lamina contains a fixed triangle ABC and a fixed point P. On the left, the trajectories generated by these points for a motion of the lamina without making a full revolution ($n = 0$). On the right, the same for another motion making a full revolution ($n = 1$).

Theorem 2.7 (Leudesdorf)**.** *Consider the motion of a lamina, on which A, B, C (all distinct) and P are fixed points, in such a way that these four points describe closed curves. Let x, y, z be the barycentric (or areal) coordinates of P referred to ABC as triangle of reference (i.e. $x + y + z = 1$). Then, it holds the relation*

$$\tilde{\mathcal{A}}(P) = x\,\tilde{\mathcal{A}}(A) + y\,\tilde{\mathcal{A}}(B) + z\,\tilde{\mathcal{A}}(C) + n\,\pi\,\mathcal{T},$$

where n is the number of complete rotations of the lamina and $\mathcal{T}$ represents the power of the point P to the circle rounding ABC.

Proof. Name $A = A(t) = \big(p_1(t), q_1(t)\big)$, $B = B(t) = \big(p_2(t), q_2(t)\big)$, $C = C(t) = \big(p_3(t), q_3(t)\big)$, $P = P(t) = \big(p(t), q(t)\big)$, $BC = a$, $AC = b$ and $AB = c$ (see Figure 2.9). For every position of the lamina, we have

$$P(t) = x\,A(t) + y\,B(t) + z\,C(t).$$

Using that $x + y + z = 1$, it is straightforward to show that

$$\begin{aligned} p(t)\,q'(t) &= (x\,p_1 + y\,p_2 + z\,p_3)\,(x\,q_1' + y\,q_2' + z\,q_3') \\ &= x\,p_1\,q_1' + y\,p_2\,q_2' + z\,p_3\,q_3' - y\,z\,(p_2 - p_3)\,(q_2' - q_3') \\ &\quad - x\,z\,(p_3 - p_1)\,(q_3' - q_1') - x\,y\,(p_1 - p_2)\,(q_1' - q_2'). \end{aligned}$$

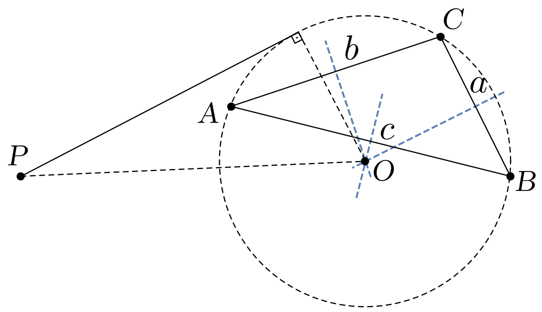

Figure 2.9: Three distinct points A, B and C and a point P in a lamina. The point O is the center of the circumcircle of the triangle ABC. The power of P to the circumcircle is the squared tangent distance if P lies outside such a circle.

where $p_i = p_i(t)$ and $q_i = q_i(t)$ for $i = 1, 2, 3$. Now, integrating the previous expression on a domain I such that the four points describe closed curves, we get

$$\tilde{\mathcal{A}}(P) = x\,\tilde{\mathcal{A}}(A) + y\,\tilde{\mathcal{A}}(B) + z\,\tilde{\mathcal{A}}(C) - y\,z \int_I (p_2 - p_3)\,(q_2 - q_3)'\,\mathrm{d}t$$
$$- x\,z \int_I (p_3 - p_1)\,(q_3 - q_1)'\,\mathrm{d}t - x\,y \int_I (p_1 - p_2)\,(q_1 - q_2)'\,\mathrm{d}t.$$

The first integral is the area traced out by C relatively to B which, since the distance between both points is always equal to a, has a value $n\,\pi\,a^2$ if the system makes n complete rotations (assuming $n = 0$ if the system makes a partial rotation and a return). Analogously for the other two integrals. Therefore,

$$\tilde{\mathcal{A}}(P) = x\,\tilde{\mathcal{A}}(A) + y\,\tilde{\mathcal{A}}(B) + z\,\tilde{\mathcal{A}}(C) - n\,\pi\left(a^2 y\,z + b^2 x\,z + c^2 x\,y\right). \quad (2.10)$$

Recall that the length of any displacement vector $\overrightarrow{RS}$ with barycentric coordinates $[\bar{x}, \bar{y}, \bar{z}]$ on the triangle ABC can be computed as

$$\left\|\overrightarrow{RS}\right\|^2 = -a^2 \bar{y}\,\bar{z} - b^2 \bar{x}\,\bar{z} - c^2 \bar{x}\,\bar{y}. \quad (2.11)$$

Now, let the origin O be placed on the circumcenter of the triangle (the intersection of the perpendicular bisectors of the sides of the triangle). By definition, the power of the point P to the circumcircle is

$$\mathcal{T} = \left\|\overrightarrow{OP}\right\|^2 - r^2,$$

where r is the radius of the circumcircle, which can be computed as $\left\|\overrightarrow{OC}\right\|$, for instance (see Figure 2.9). Hence, since $\overrightarrow{OP} = [x, y, z]$ and $\overrightarrow{OC} = [0, 0, 1]$, using (2.11), we have

$$\mathcal{T} = \left\|\overrightarrow{OP}\right\|^2 - \left\|\overrightarrow{OC}\right\|^2 = -a^2\,y\,z - b^2\,x\,z - c^2\,x\,y.$$

The result follows by substitution on (2.10). $\square$

Remark 2.8. As a particular case of Theorem 2.7, if P lies in the segment AB so that it divides that chord into two lengths p and q, we have that the barycentric coordinates of P are $[q/(p+q), p/(p+q), 0]$. Moreover, the power of the point P to the circumcircle of the triangle ABC can be computed with the line that passes through P containing AB (Theorem 2.6), so that it is equal to $-p\,q$. Therefore, Leudesdorf's formula takes the form

$$\tilde{\mathcal{A}}(P) = \frac{q\,\tilde{\mathcal{A}}(A) + p\,\tilde{\mathcal{A}}(B)}{p+q} - n\,\pi\,p\,q,$$

which is Woolhouse's extension of Holditch's theorem (Theorem 2.2).

In [44], Kempe made some remarks about the original Leudesdorf's theorem. Among other things, it mentions that it is not necessary that the curves traced out by the points A, B, C and P should all be closed curves, instead, it is enough if the points return to their starting places.

As a remark, in the construction of kinematics given in Edwards' book [24], Leudesdorf's theorem is proved as a consequence of Woolhouse's.

As a corollary of Theorem 2.7 a result due to Kempe can be deduced. Different proofs can be found in the literature. The proof given below follows the idea of [97] but making it simpler and more detailed. Before that, it is worth to give a preliminary version.

Lemma 2.9. *If one plane sliding upon another start from any position, move in any manner, and return to its original position after making one or more complete revolutions; then, every point in the moving plane describes a closed curve in the fixed plane. If two distinct points on the moving plane describe the same area in the fixed one, then there is another one describing the same area.*

Proof. If there are two distinct points A and B describing the same area $\tilde{\mathcal{A}}(A)$, it is easy to show that there are in fact two more describing such an area. Indeed, if A and B are at a distance $2\,r$, consider a point C at the same distance $\sqrt{2}\,r$ from both points A and B (see Figure 2.10). A point X in the bisector of AB at a distance λ from the circumcenter of ABC has barycentric coordinates:

$$\left[\frac{r-\lambda}{2\,r}, \frac{r-\lambda}{2\,r}, \frac{\lambda}{r}\right].$$

By Leudesdorf's theorem:

$$\tilde{\mathcal{A}}(X) = \frac{r-\lambda}{r}\,\tilde{\mathcal{A}}(A) + \frac{\lambda}{r}\,\tilde{\mathcal{A}}(C) - n\,\pi\,(\lambda^2 - r^2),$$

where n is the number of revolutions. Impose that $\tilde{\mathcal{A}}(X) = \tilde{\mathcal{A}}(A)$. Thus,

$$0 = \lambda\left(\tilde{\mathcal{A}}(C) - \tilde{\mathcal{A}}(A)\right) - n\,\pi\,r\,(\lambda^2 - r^2).$$

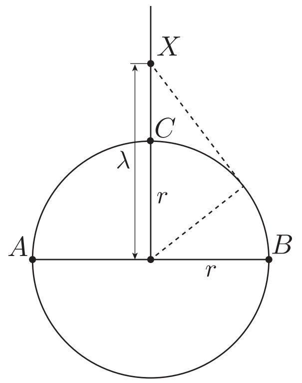

Figure 2.10: Two distinct points A and B a distance $2\,r$ apart. The point C is at a distance $\sqrt{2}\,r$ from A and B. We are looking for a point X in the bisector of AB such that the area described by X is the same as the one described by A or B.

Solving this equation for λ, two real solutions are found, which lead to two possible points X such that $\tilde{\mathcal{A}}(X) = \tilde{\mathcal{A}}(A)$. $\square$

Kempe's theorem states that in the setting of Lemma 2.9, there are, in fact, infinite points describing the same area and such points form a circle.

Theorem 2.10 (Kempe)**.** *If one plane sliding upon another start from any position, move in any manner, and return to its original position after making one or more complete revolutions; then, every point in the moving plane describes a closed curve in the fixed plane, and the locus, in the moving plane, of points which describe equal areas (in the fixed one) is a circle; and by varying the area a system of concentric circles for loci is found.*

Proof. If there is only one point describing an area $\tilde{\mathcal{A}}(A)$, then it can be seen as a zero-circle in the moving plane. Otherwise, by Lemma 2.9, let A, B and C be three distinct points which describe an equal area: $\tilde{\mathcal{A}}(A)$. Also, let P and P' be two distinct points describing an area $\tilde{\mathcal{A}}(P)$. By Leudesdorf's theorem,

$$\tilde{\mathcal{A}}(P) = \tilde{\mathcal{A}}(A) + n\,\pi\,\mathcal{T}_P,$$

where $\mathcal{T}_P$ is the power of P to the circumcircle of ABC, which must equal the power of P' to the same circumcircle, $\mathcal{T}_{P'}$. If O is the center and r is the radius of such a circle, by definition of the power of a point, $OP = OP'$, which proves that P and P' lie on the same circle, which is centered at O. Of course, varying the area $\tilde{\mathcal{A}}(P)$, the power of the tangent from the point to the circumcircle will vary and so the radius of the circle centered at O, which leads to the loci of concentric circles. $\square$

The original statement of Kempe's theorem can be found in [45] and its proof in [47]. It was also published in [43]. See [46] for two little remarks about the theorem.

Remark 2.11. Regarding Theorem 2.10, as stated originally by Kempe, if the moving plane returns to its initial position without having made a complete rotation, the system of concentric circles is replaced by a system of parallel straight lines.

2.4 Elliott's extensions of Holditch's theorem

2.4.1 A variable length with a constant ratio

In [26], Elliott gave an extension of Woolhouse's theorem by supposing a variable length in the moving chord such that the ratio $m : n$ in which the tracing point divides the chord is constant. The proof we will give here can be found in the original paper by Elliott or in Williamson's calculus [97].

Theorem 2.12 (Elliott)**.** *Let $\delta, \alpha : I \to \mathbb{R}^2$ be two closed planar curves and let P be a point in the closed region determined by δ. Consider all the radii vectors from P to $\delta(s)$ and let them be placed with one extremity in $\alpha(s)$, so the other extremity will be placed in another closed curve $\beta(s)$ (see Figure 2.11). If both extremities travel all around the perimeters and do not return to their first positions from the same sides as that towards which they left them; and if γ is the curve generated by a point dividing the radii vectors in a constant ratio $m(s) : n(s)$, then we have that*

$$\tilde{\mathcal{A}}(\gamma) = \frac{n(s)\,\tilde{\mathcal{A}}(\alpha) + m(s)\,\tilde{\mathcal{A}}(\beta)}{m(s)+n(s)} - \frac{m(s)\,n(s)}{\big(m(s)+n(s)\big)^2}\,\tilde{\mathcal{A}}(\delta). \tag{2.12}$$

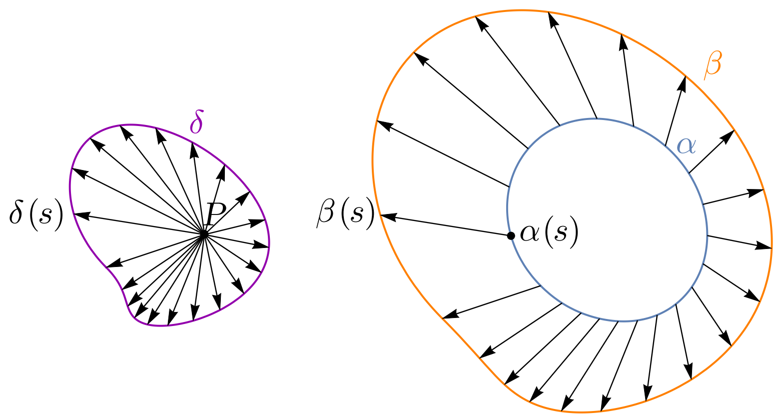

Figure 2.11: On the left, the curve δ of radii vectors. On the right, a generated curve β from the initial α with those vectors.

Proof. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be the coordinates of the curves α and β, respectively. Let $\lambda = m(s)/n(s)$ be the constant ratio. The generated curve γ of the statement can be written as

$$\gamma(s) = \frac{n(s)\,\alpha(s) + m(s)\,\beta(s)}{m(s)+n(s)} = \frac{\alpha(s) + \lambda\,\beta(s)}{\lambda+1},$$

so $(\lambda+1)\,\gamma(s) = \alpha(s) + \lambda\,\beta(s)$. If $\gamma = (\gamma_1, \gamma_2)$, then we can compute

$$(\lambda+1)^2\,\gamma_1\,\gamma_2' = \alpha_1\alpha_2' + \lambda^2\,\beta_1\,\beta_2' + \lambda\,(\alpha_1\,\beta_2' + \alpha_2'\,\beta_1), \tag{2.13}$$

so, integrating we get

$$(\lambda+1)^2\tilde{\mathcal{A}}(\gamma) = \tilde{\mathcal{A}}(\alpha) + \lambda^2\,\tilde{\mathcal{A}}(\beta) + \lambda \int_I \big(\alpha_1(s)\,\beta_2'(s) + \alpha_2'(s)\,\beta_1(s)\big)\;\mathrm{d}s. \tag{2.14}$$

Now, since

$$\int_I \big(\alpha_1(s)\,\beta_2'(s) + \alpha_2'(s)\,\beta_1(s)\big)\;\mathrm{d}s \\ = \tilde{\mathcal{A}}(\alpha) + \tilde{\mathcal{A}}(\beta) - \int_I \big(\beta_1(s)-\alpha_1(s)\big)\big(\beta_2(s)-\alpha_2(s)\big)'\;\mathrm{d}s,$$

then we can write (2.14) as

$$(\lambda+1)^2\tilde{\mathcal{A}}(\gamma) = (\lambda+1)\,\tilde{\mathcal{A}}(\alpha) + \lambda\,(\lambda+1)\,\tilde{\mathcal{A}}(\beta) \\ - \lambda \int_I \big(\beta_1(s)-\alpha_1(s)\big)\big(\beta_2(s)-\alpha_2(s)\big)'\;\mathrm{d}s,$$

from which we get

$$\tilde{\mathcal{A}}(\gamma) = \frac{\tilde{\mathcal{A}}(\alpha) + \lambda\,\tilde{\mathcal{A}}(\beta)}{\lambda+1} - \frac{\lambda}{(\lambda+1)^2}\,\tilde{\mathcal{A}}(\delta).$$

Note that this expression yields the result, so (2.12) does not depend on the functions $m(s)$ and $n(s)$, only on its constant ratio $\lambda = m(s)/n(s)$. □

Remark 2.13. In the original paper, the area $\tilde{\mathcal{A}}(\delta)$ of Theorem 2.12 is referred to as a relative area. Indeed, as it is seen in the proof, it is the signed area of the curve $\beta(s)-\alpha(s)$, so it can be expressed as

$$\int_I \big(\beta_1(s)-\alpha_1(s)\big)\big(\beta_2(s)-\alpha_2(s)\big)'\;\mathrm{d}s,$$

with $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$.

The idea behind the proof of Theorem 2.12 is to write the expression to integrate, Equation (2.13), as a sum of three volume elements:

$$(\lambda+1)^2\,\gamma_1\,d\gamma_2 = (\lambda+1)\,\alpha_1\,d\alpha_2 + \lambda\,(\lambda+1)\,\beta_1\,d\beta_2 - \lambda\,(\beta_1-\alpha_1)\,(d\beta_2 - d\alpha_2),$$

and then integrate. This idea will be essential for the proof of Theorem 2.15, a generalization of this result to three dimensions.

Remark 2.14. See in Figure 2.12 an example in the setting of Theorem 2.12. Now, from that result some particular cases can be pointed out. Let's deduce from it Theorem 2.2 by Woolhouse. If the length $m(s)+n(s) = \ell$ is constant, then since $m(s) = \lambda\,n(s)$, we get that

$$n(s) = \frac{\ell}{\lambda+1} \quad \text{and} \quad m(s) = \frac{\lambda\,\ell}{\lambda+1}$$

are both constants; say $m(s) = p$ and $n(s) = \ell - m(s) = \ell - p$. Moreover, since chord length is constant and equal to ℓ, the relative area δ is described by

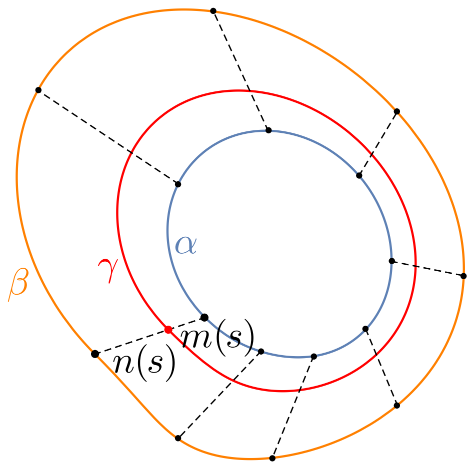

Figure 2.12: An example of the motion of a chord of variable length between two initial curves, α and β, and a curve γ generated by a point dividing such a chord into a constant ratio $\lambda = \frac{m(s)}{n(s)} = \frac{1}{2}$.

a circle of radius ℓ. Assuming the moving chord to rotate counterclockwise, such an area is equal to $\pi\,\ell^2$. Therefore, from (2.12) we get

$$\tilde{\mathcal{A}}(\gamma) = \frac{(\ell - p)\,\tilde{\mathcal{A}}(\alpha) + p\,\tilde{\mathcal{A}}(\beta)}{\ell} - \pi\,p\,(\ell - p).$$

For more applications and particular cases, see the cited paper [26]. For a proof of Theorem 2.12 using polar coordinates see [23].

2.4.2 Holditch's theorem for volumes

In the paper of Broman, [11], there is a little remark about the non-existence of some kind of generalization of Holditch's theorem in $\mathbb{R}^3$ talking about volumes instead of areas. With an example, Broman sees that there are some examples where the result does not only depends on the lengths in which the point divides the chord, so the hoped result does not hold. Nevertheless, it is actually possible to find some generalizations—but not that simple—in the case of volumes as Elliott showed in its paper [26]. Even more, the result is given in terms of constant ratios as in Theorem 2.12.

Theorem 2.15 (Elliott)**.** *Let $\nu, \alpha : I \to \mathbb{R}^3$ be two closed volumes in $\mathbb{R}^3$ and let P be a point in the closed volume determined by ν. Consider all the radii vectors from P to the boundary of ν and let them be placed with one extremity in α, so the other extremity will be placed in another closed volume β (see Figure 2.13). If both extremities travel all around the surfaces smoothly such that rounded positions of α corresponds to rounded positions of β; and if γ is the curve generated by a point dividing the radii vectors in a constant ratio*

$m(s) : n(s)$, then we have that

$$\begin{aligned}\mathcal{V}(\gamma) &= \frac{m(s)-n(s)}{\big(m(s)+n(s)\big)^2}\big(m(s)\,\mathcal{V}(\beta)-n(s)\,\mathcal{V}(\alpha)\big)\\ &\quad - \frac{m(s)\,n(s)\,\big(m(s)-n(s)\big)}{2\,\big(m(s)+n(s)\big)^3}\,\mathcal{V}(\nu)+\frac{4\,m(s)\,n(s)}{\big(m(s)+n(s)\big)^2}\,\mathcal{V}(\mu),\end{aligned} \tag{2.15}$$

where μ is the locus of the middle point of the rod.

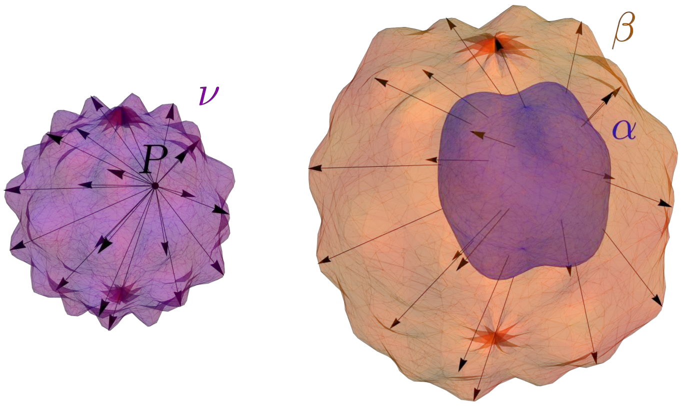

Figure 2.13: On the left, the solid volume ν of radii vectors. On the right, a generated volume β from the initial α with those vectors.

Proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ be the coordinates of the volumes α and β, respectively. The generated volume of the statement, γ, can be defined by

$$\gamma = \frac{n\,\alpha + m\,\beta}{m+n} = \frac{\alpha + \lambda\,\beta}{\lambda + 1}$$

for the constant ratio $\lambda = \frac{m(s)}{n(s)}$, so

$$(\lambda+1)\gamma = \alpha + \lambda\,\beta.$$

If $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, then we can compute

$$(\lambda+1)^3\gamma_3\,d\gamma_1\,d\gamma_2 = (\alpha_3 + \lambda\,\beta_3)\,(d\alpha_1 + \lambda\,d\beta_1)\,(d\alpha_2 + \lambda\,d\beta_2).$$

We can write this expression as a sum of four volume elements as follows:

$$\begin{aligned}(\lambda+1)^3\gamma_3\,d\gamma_1\,d\gamma_2 &= \lambda\,(\lambda^2-1)\,\beta_3\,d\beta_1\,d\beta_2 - (\lambda^2-1)\,\alpha_3\,d\alpha_1\,d\alpha_2\\ &\quad - \frac{\lambda\,(\lambda-1)}{2}\,(\beta_3-\alpha_3)\,(d\beta_1-d\alpha_1)\,(d\beta_2-d\alpha_2)\\ &\quad + \frac{\lambda\,(\lambda+1)}{2}\,(\beta_3+\alpha_3)\,(d\beta_1+d\alpha_1)\,(d\beta_2+d\alpha_2).\end{aligned}$$

Now, integrating, we get

$$\begin{aligned}(\lambda+1)^3\mathcal{V}(\gamma) &= \lambda\,(\lambda^2-1)\,\mathcal{V}(\beta)-(\lambda^2-1)\,\mathcal{V}(\alpha)\\ &\quad-\frac{\lambda\,(\lambda-1)}{2}\,\mathcal{V}(\nu)+4\,\lambda\,(\lambda+1)\,\mathcal{V}(\mu).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{V}(\gamma) &= \frac{\lambda-1}{(\lambda+1)^2}\big(\lambda\,\mathcal{V}(\beta)-\mathcal{V}(\alpha)\big)\\ &\quad-\frac{\lambda\,(\lambda-1)}{2\,(\lambda+1)^3}\,\mathcal{V}(\nu)+4\,\frac{\lambda}{(\lambda+1)^2}\,\mathcal{V}(\mu).\end{aligned}$$

This expression does not depend on the functions $m(s)$ and $n(s)$, only on its constant ratio λ. Writing it in terms of $m(s)$ and $n(s)$, Equation (2.15) of the statement is deduced. □

Remark 2.16 (A particular case)**.** If the rod of Theorem 2.15 is of constant length $m+n$ such that γ divides it into two constant lengths m and n, then we have that ν becomes a sphere, so $\mathcal{V}(\nu)=\frac{4}{3}\,\pi\,(m+n)^3$. In this particular case, the formula (2.15) takes the form

$$\begin{aligned}\mathcal{V}(\gamma) &= \frac{m-n}{(m+n)^2}\big(m\,\mathcal{V}(\beta)-n\,\mathcal{V}(\alpha)\big)\\ &\quad-\frac{2\pi}{3}\,m\,n\,(m-n)+\frac{4\,m\,n}{(m+n)^2}\,\mathcal{V}(\mu).\end{aligned}$$

Remark 2.17. The relation of Theorem 2.15 can be seen as a connection of the volumes corresponding to four points varying on a straight line, one of them being the middle point of the segment delimited by two of the others. Moreover, a general relation with four points on the line avoiding that restriction can be found. The interested reader can see [26] for more details.

2.4.3 Extension to areas of curved surfaces

In the paper [28], Elliott applied its extension of Holditch's theorem (Theorem 2.12) to obtain a generalized version on areas in some curved surfaces. The result together with the original proof of his paper can be written as follows.

Theorem 2.18 (Elliott)**.** *Let A and B be two continuous portions of curved surfaces such that every point of A corresponds to a point of B and inversely such that both regions are completely covered, the boundaries correspond together and the tangent planes at pairs of any corresponding points are parallel (see Figure 2.14). Let C be the region of the curved surface covered by the points dividing the chords from A to B in a constant ratio $m:n$. Then, we have the relation*

$$\tilde{\mathcal{A}}(C)=\frac{n\,\tilde{\mathcal{A}}(A)+m\,\tilde{\mathcal{A}}(B)}{m+n}-\frac{m\,n}{\big(m+n\big)^2}\,\tilde{\mathcal{A}}(S), \tag{2.16}$$

where S is the curved surface generated by parallel and equal vectors to all the chords of corresponding points from A to B but drawn from a fixed point (surface of radii vectors).

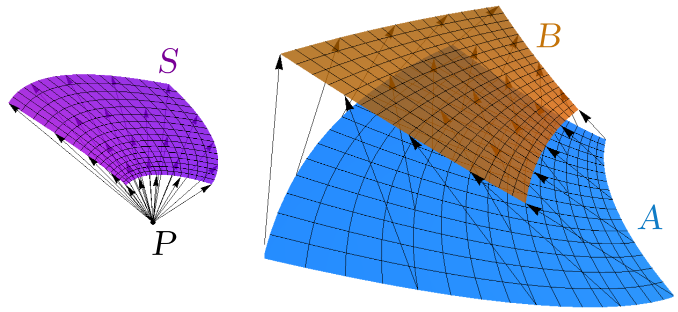

Figure 2.14: On the left, the surface S of radii vectors. On the right, a generated surface B from the initial A with those vectors. The surfaces A and B are such that the corresponding tangent planes are parallel.

Proof. Three elementary facts are going to be used:

(i) If we have three points in each of two parallel planes, then the points in which the chords joining them two by two are divided in any same ratio lie in another parallel plane.

(ii) In the same case, if from any fixed point lines are drawn parallel and equal to that chords, their other extremities lie in a plane parallel to each of both planes.

(iii) Any orthogonal projection of a divided chord is a similarly divided chord in the projection.

If A and B generate areas in different parallel planes, the locus C of the points dividing the chords joining points of A with points of B in the same ratio $m : n$ is a region in a plane parallel to those of A and B by (i). By (ii), the relative locus of the points of B with regard to A is also a region S in another parallel plane. Projecting the system orthogonally upon any plane, we have a relation like (2.16) by Theorem 2.12 but for the projected areas and lengths. By the similarity stated in (iii), we obtain the same relation for non-projected lengths.

Finally, in particular, the previous relation is true for infinitesimal areas:

$$\tilde{A}(\delta C) = \frac{n\,\tilde{A}(\delta A) + m\,\tilde{A}(\delta B)}{m+n} - \frac{m\,n}{(m+n)^2}\,\tilde{A}(\delta S)$$

Making up portions of curved surfaces A and B with contiguous infinitesimal areas δA and δB, respectively, then by the stated facts we will have continuous curved surfaces C and S made up by the elements δC and δS, respectively. Integrating the system the desired result is obtained. □

2.5 Kinematics on a sphere

As showed in the previous sections, Elliott did a great contribution on generalizations of Holditch's theorem. In a paper of 1881, [27], Elliott presented his study of the analogue in the sphere to the theorems of Holditch, Leudesdorf and Kempe. The idea is to study kinematics in the sphere, specifically, relations with spherical areas determined by spherical figures that move upon a sphere without changing their size or form ending in the original position. In fact, the main theorem of the cited paper is a generalization of the result given by Gaston Darboux in [18].

In the following, the approach of Elliott to the problem of kinematics in the sphere is presented. Finally, in Proposition 2.19 a particular case of a Holditch motion with two initial curves is considered.

Consider in $\mathbb{R}^3$ the sphere of radius R centered at the origin O. Take a geodesic chord between two points A and B in that sphere such that subtends an angle $\alpha+\beta$ at the sphere's centre. Consider a point C dividing this arc into two constant parts α and β (see Figure 2.15).

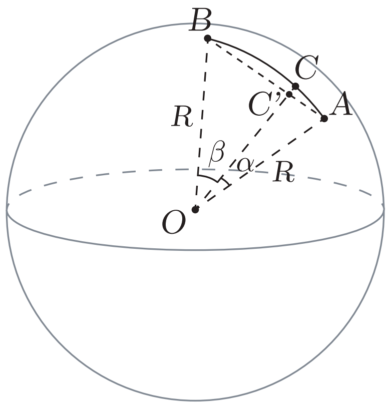

Figure 2.15: A sphere of radius R and a geodesic chord of constant length with its two ends A and B on the sphere. A point C divides such a chord into two arcs of angles α and β.

The arc joining A and B is of constant length $\ell = R\,(\alpha+\beta)$. It is also of constant length, say ℓ', the straight segment joining A and B; by the cosine law,

$$\ell'^2 = 2\,R^2\big(1-\cos(\alpha+\beta)\big) = 4\,R^2\sin^2\left(\frac{\alpha+\beta}{2}\right),$$

so that

$$\ell' = 2\,R\sin\left(\frac{\alpha+\beta}{2}\right).$$

Moreover, if C' is the intersection point between this straight segment and the radius of the sphere joining C, then by the sine laws,

$$\frac{AC'}{\sin\alpha} = \frac{R}{\sin\angle(OC'A)} \quad \text{and} \quad \frac{C'B}{\sin\beta} = \frac{R}{\sin\angle(BC'O)}.$$

Therefore, since $\sin\angle(BC'O) = \sin\big(\pi - \angle(OC'A)\big) = \sin\angle(OC'A)$, it is deduced that

$$\frac{AC'}{C'B} = \frac{\sin\alpha}{\sin\beta}, \tag{2.17}$$

which means that the straight segment is divided by C' in a constant ratio. Thus, the problem of a moving geodesic chord on the sphere and a point C dividing it into two constant parts can be reduced to the case of a moving straight segment of constant length with its endpoints on the sphere and the point C' dividing it in a constant ratio.

Again, by the sine law,

$$\frac{OC'}{\sin\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right)} = \frac{R}{\sin\angle(OC'A)},$$

where

$$\angle(OC'A) = \pi - \alpha - \frac{1}{2}\left(\pi - (\alpha+\beta)\right) = \frac{\pi}{2} + \frac{\beta-\alpha}{2}.$$

From this,

$$OC' = R\,\frac{\sin\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\pi}{2} + \frac{\beta-\alpha}{2}\right)} = R\,\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)},$$

which is the radius of the concentric sphere on which C' moves.

With the previous idea and setting some kinematics theorems can be deduced, the interested reader can look at [27]. Below it is only stated and proved the first elementary result following the original proof by Elliott.

Proposition 2.19. *Let a geodesic arc AB subtending a constant angle $\alpha+\beta$ be divided by a point C into two parts of angles α and β, respectively, move on a sphere of radius R. If each endpoint A and B of such a chord describe a closed curve (and so C also do) such that their spherical areas can be covered by pairs of corresponding endpoints of the chord without making a full revolution, then*

$$\tilde{\mathcal{A}}(C) = \frac{\sin\beta\,\tilde{\mathcal{A}}(A) + \sin\alpha\,\tilde{\mathcal{A}}(B)}{\sin(\alpha+\beta)}.$$

Proof. The idea is to compute areas infinitesimally. Given the rod AB, three distinct kinds of motions are considered:

1. a rotation about the centre of the sphere in the great circle plane π_0 which contains it;

2. a rotation about its own middle point in the plane which contains it and is at right angles to π_0; and

3. a translation at right angles to itself in this second plane.

For an infinitesimal displacement $d\theta$ of the first kind, the points A and B are moved through distances $R\,d\theta$; and the point C' through a distance

$$R\,\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\,d\theta.$$

Let $c = \ell'/2$. For simultaneously or successively displacements $d\psi$, ds of the second and third kind, respectively; the points A and B are displaced $ds + c\,d\psi$ and $ds - c\,d\psi$, respectively, in directions at right angles to the first displacements of those points, and that of C' upon its own sphere will be

$$ds + c\,\frac{\sin\beta - \sin\alpha}{\sin\alpha + \sin\beta}\,d\psi$$

at right angles to its first displacement. Of course, note that

$$c' := c\,\frac{\sin\beta - \sin\alpha}{\sin\alpha + \sin\beta} \tag{2.18}$$

is the radius of the circle, contained in the second plane, centered at the middle point of the rod and that passes through C' (see Figure 2.16). Indeed, the point C' divides the rod AB into two parts AC' and $C'B$ such that $AC' + C'B = 2\,c$ and (2.17). With that, it can be deduced that

$$AC' = 2\,c\,\frac{\sin\alpha}{\sin\alpha + \sin\beta}, \quad C'B = 2\,c\,\frac{\sin\beta}{\sin\alpha + \sin\beta}.$$

If $\beta < \alpha$, then the radius is $\frac{1}{2}\,(C'B - AC')$. Otherwise, the radius is $\frac{1}{2}\,(AC' - C'B)$, but in this case the rotation must be taken with a minus sign. In any case, such a radius equals (2.18).

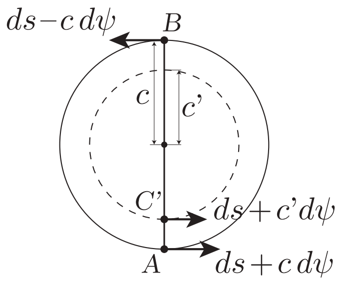

Figure 2.16: Projection onto the plane which contains the middle point of the rod and is orthogonal to π_0. The displacements of the second and third kind are shown for the points A, B and C'.

Now, the rectangular elements of area in the given sphere contained by these rectangular displacements of A and B are

$$dA = R\,d\theta\,(ds + c\,d\psi) \tag{2.19}$$

and

$$dB = R\, d\theta\, (ds - c\, d\psi), \tag{2.20}$$

respectively. Also, the rectangular element on the sphere of C' is

$$dC' = R\, \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\, d\theta \left(ds + c\, \frac{\sin\beta - \sin\alpha}{\sin\alpha + \sin\beta}\, d\psi\right). \tag{2.21}$$

From (2.19) and (2.20),

$$d\theta\, ds = \frac{dA + dB}{2\, R}, \quad \text{and} \quad d\theta\, d\psi = \frac{dA - dB}{2\, R\, c}.$$

Substituting these expressions in (2.21) and arranging, it is found:

$$dC' = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\, \frac{\sin\beta\, dA + \sin\alpha\, dB}{\sin\alpha + \sin\beta}.$$

Since both endpoints A and B describe a closed curve on the sphere, the point C' will also do on its own sphere. By hypothesis, the area elements dA, dB and dC' cover their full corresponding areas, so by integration on such regions,

$$\tilde{\mathcal{A}}(C') = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\, \frac{\sin\beta\, \tilde{\mathcal{A}}(A) + \sin\alpha\, \tilde{\mathcal{A}}(B)}{\sin\alpha + \sin\beta}.$$

To get the previous expression in terms of $\tilde{\mathcal{A}}(C)$ instead of $\tilde{\mathcal{A}}(C')$, it must only be noticed that

$$dC = \frac{\cos^2\left(\frac{\alpha-\beta}{2}\right)}{\cos^2\left(\frac{\alpha+\beta}{2}\right)}\, dC'.$$

With that, the corresponding conclusion is

$$\tilde{\mathcal{A}}(C) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\, \frac{\sin\beta\, \tilde{\mathcal{A}}(A) + \sin\alpha\, \tilde{\mathcal{A}}(B)}{\sin\alpha + \sin\beta} = \frac{\sin\beta\, \tilde{\mathcal{A}}(A) + \sin\alpha\, \tilde{\mathcal{A}}(B)}{\sin(\alpha+\beta)}.$$

□

The previous result is extended along the Elliott's paper for any number n of chord revolutions, in such a way that the following formula holds:

$$\tilde{\mathcal{A}}(C) = \frac{\sin\beta\, \tilde{\mathcal{A}}(A) + \sin\alpha\, \tilde{\mathcal{A}}(B)}{\sin(\alpha+\beta)} - 4\, n\, \pi\, R^2\, \frac{\sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}.$$

This expression is, in fact, the extension of Woolhouse's theorem to the sphere. This result and some others to any constant curvature surface will be addressed in Chapter 5.

Chapter 3

Moving chords in the plane

In this chapter the Holditch setting is presented: the Holditch functions and the Holditch curves are formally defined and a result on the existence of these curves is given. In addition, some properties for the non-retrograde motion case are derived. Later, a generalization of Holditch's theorem to polygons is presented with the aim to glimpse the ellipse from which the Holditch area comes. To do it properly, the Holditch map is defined and its continuity is studied. Afterwards, a general kind of curves is considered (including parallel curves, constant width curves and Holditch curves) and some results involving areas and lengths of these curves are deduced.

3.1 Definition of curves in a Holditch motion

3.1.1 Definition of Holditch functions

Let $\alpha : I \to \mathbb{R}^2$ be a simple planar curve, where I is some interval and $\ell > 0$ is the length of the moving chord. The moving chord is sometimes called the *Holditch chord*. In this section, the functions which describe the movement of the endpoints of a moving chord will be defined.

Forward movement of the chord

Given an initial position of the moving chord, assume that it can travel smoothly and always forwards along α (according to the orientation of α). If $\alpha(s)$ describes the rear endpoint of the chord, then the front endpoint can be described by means of some injective continuous function f as $\alpha\big(f(s)\big)$. This function is defined by the following restriction on the constant length of the chord:

$$\Big\|\alpha\big(f(s)\big) - \alpha(s)\Big\| = \ell.$$

Therefore, notice that f depends on the length ℓ and also on the parameterization α of the initial curve. The function f is called the *Holditch function for the parameterization α and the chord length ℓ* (see Figure 3.1).

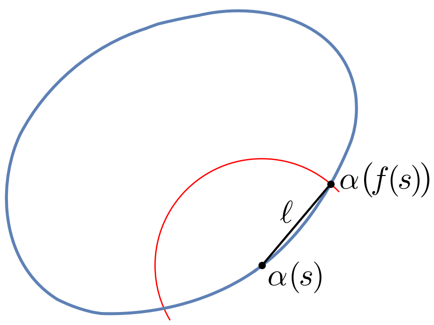

Figure 3.1: Definition of the Holditch function f in a smooth and always forward travel of the moving chord.

Define

$$R := \begin{cases} I & \text{if } \alpha \text{ is open,} \\ \mathbb{R} & \text{if } \alpha \text{ is closed,} \end{cases}$$

and

$$J_\ell := \Big\{ s \in I : \exists\, t \in R,\ t \geq s,\ \big\|\alpha(t) - \alpha(s)\big\| = \ell \Big\}.$$

If α is open, then $J_\ell \subseteq I$ and also $f(J_\ell) \subseteq I$. Thus, the Holditch function $f : J_\ell \to f(J_\ell)$ is a well-defined homeomorphism (see Figure 3.2). If α is closed, $J_\ell = I$ and $f(I) \subseteq \mathbb{R}$. Thus, the Holditch function $f : I \to f(I)$ is

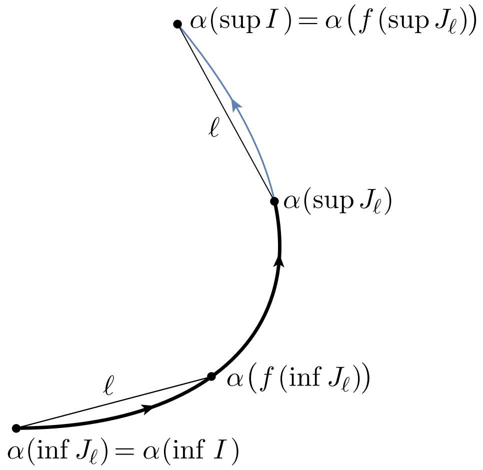

Figure 3.2: In an open curve $\alpha : I \to \mathbb{R}^2$ the Holditch function f for a chord length ℓ has domain $J_\ell \subseteq I$.

also a well-defined homeomorphism (recall that the parameterization α can be extended to the whole $\mathbb{R}$ as commented in Section 1.1.2).

Thus, the Holditch constraint can be written as follows:

$$\boxed{\Big\|\alpha\big(f(s)\big) - \alpha(s)\Big\| = \ell \text{ for all } s \in J_\ell.} \tag{3.1}$$

Example 3.1. Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be the parameterized *circle*

$$\alpha(s) = r\,\big(\cos(s),\, \sin(s)\big)$$

of radius $r > 0$, where the injectivity interval is $I = [0, 2\pi[$. Given a moving chord of length $0 < \ell < 2\,r$, the objective is to compute the Holditch function f for the parameterization α.

The direct way is to put

$$\alpha\big(f(s)\big) - \alpha(s) = r\,\Big(\cos\big(f(s)\big) - \cos(s),\, \sin\big(f(s)\big) - \sin(s)\Big)$$

and to compute

$$\ell^2 = \Big\|\alpha\big(f(s)\big) - \alpha(s)\Big\|^2 = 2\,r^2\,\Big(1 - \cos\big(f(s) - s\big)\Big).$$

From there, the Holditch function $f : [0, 2\pi[\to f\big([0, 2\pi[\big)$ is

$$f(s) = s + \arccos\left(1 - \frac{\ell^2}{2\,r^2}\right). \tag{3.2}$$

A more geometric construction of f is also possible in this example (see Figure 3.3). Given $s_0 \in [0, 2\pi[$, from the law of sines,

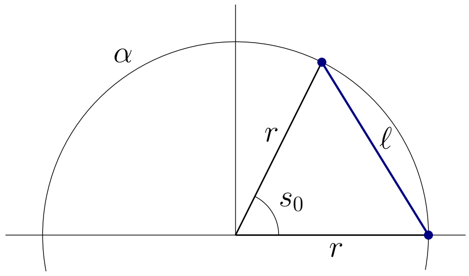

Figure 3.3: Geometric deduction of the Holditch function for a circle of radius r and a moving chord of length ℓ.

$$\frac{\ell}{\sin(s_0)} = \frac{r}{\sin\left(\frac{\pi - s_0}{2}\right)}.$$

Equivalently,

$$\ell \cos\left(\frac{s_0}{2}\right) = r \sin(s_0).$$

Now, since

$$\sin(s_0) = 2 \sin\left(\frac{s_0}{2}\right) \cos\left(\frac{s_0}{2}\right),$$

it is finally obtained (by symmetry) that the increase of angle by means of the Holditch function is

$$s_0 = 2 \arcsin\left(\frac{\ell}{2\,r}\right).$$

Therefore, the Holditch function is just the translation $f(s) = s + s_0$, which is, in fact, an equivalent expression to (3.2).

Forward and backward movement of the chord

In general, it can happen that only a forward movement of the chord is not always possible to complete a full revolution. In that case, the endpoints of the moving chord pass more than once through the same points. That phenomenon is called *retrograde motion*. In the retrograde motion case, the endpoints of the chord must be described by two functions, say g and h, instead of only one. These two functions are continuous but they are not injective. The restriction on them is the same as before:

$$\Big\| \alpha\big(h(s)\big) - \alpha\big(g(s)\big) \Big\| = \ell.$$

Sometimes, g is called the *rear Holditch function* and h the *front Holditch function* for the parameterization α and the chord length ℓ.

Definition 3.2 (Retrograde motion)**.** The moving chord is said to have *retrograde motion* if either g or h are not injective.

In this case, define

$$J_\ell^g := \Big\{ s \in I : \exists\, t \in R,\ t \geq g(s),\ \Big\| \alpha(t) - \alpha\big(g(s)\big) \Big\| = \ell \Big\}.$$

If α is open, $J_\ell^g \subseteq I$, $g\big(J_\ell^g\big) \subseteq I$ and $h\big(J_\ell^g\big) \subseteq I$. Thus, the well-defined Holditch functions are $g : J_\ell^g \to g\big(J_\ell^g\big)$ and $h : J_\ell^g \to h\big(J_\ell^g\big)$. If α is closed, $J_\ell^g = I$ and $g(I), h(I) \subseteq \mathbb{R}$. Therefore, the well-defined Holditch functions are $g : I \to g(I)$ and $h : I \to h(I)$.

Thus, the Holditch constraint in the retrograde motion case can be written as follows:

$$\boxed{\Big\| \alpha\big(h(s)\big) - \alpha\big(g(s)\big) \Big\| = \ell \text{ for all } s \in J_\ell^g.} \tag{3.3}$$

Example 3.3. In easy examples of curves, it can be seen that the retrograde motion behavior has to do with big curvature regions. A classic example for that is any equilateral triangle, in whose vertices it is concentrated the total curvature. Thus, a basic case to study corresponds to an *acute angle* formed by two rays. Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be the parameterization

$$\alpha(t) = \begin{cases} (t,\, 0), & \text{if} \quad t \leq 1, \\ (2-t,\, t-1), & \text{if} \quad t > 1. \end{cases}$$

In this case, the Holditch functions can be computed explicitly for any chord length $\ell > 0$ and they are

$$g(t) = \begin{cases} t, & \text{if} \quad t \leq 1-\ell, \\ 2-\ell-t-\sqrt{(1-t)\,(t+2\,\ell-1)}, & \text{if} \quad 1-\ell < t \leq 1, \\ t-\ell, & \text{if} \quad t > 1, \end{cases}$$

and

$$h(t) = \begin{cases} t+\ell, & \text{if} \quad t \leq 1, \\ 1+\frac{1}{2}\left(1+\ell+\sqrt{\ell^2-2\,\ell\,(1-t)-(1-t)^2}-t\right), & \text{if} \quad 1 < t \leq 1+\ell, \\ t-\ell+\frac{\ell}{\sqrt{2}}, & \text{if} \quad t > 1+\ell. \end{cases}$$

In Figure 3.4, such functions are represented for $\ell = 1/2$. Indeed, this is an example of non-injective Holditch functions, so retrograde movement happens when sliding the moving chord. Figure 3.5 shows some frames of the movement to understand its behavior.

Chords on Jordan curves

The most common case to work with in a Holditch setting corresponds to closed curves. More specifically, to Jordan curves. In such a case, there is

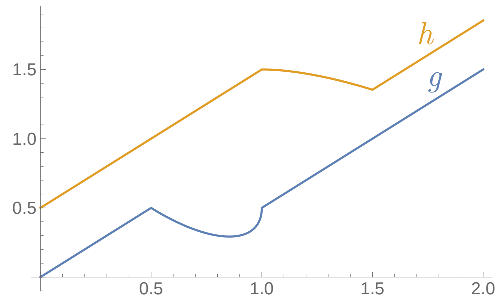

Figure 3.4: The Holditch functions in the acute angle α for $\ell = 1/2$.

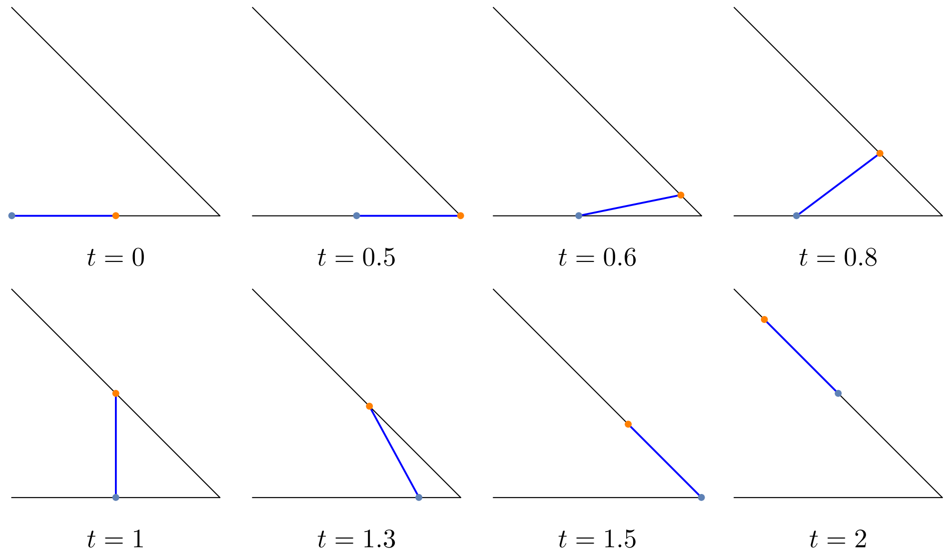

Figure 3.5: For $\ell = 1/2$, the Holditch motion frame-by-frame for different t values to show the retrograde movement phenomenon.

another way to define the initial curve and the movement of the chord, this is by means of the 1-dimensional sphere:

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Thus, the initial curve is seen as $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ and the Holditch functions as continuous maps $g, h : \mathbb{S}^1 \to \mathbb{S}^1$. The usual parameterization of $\mathbb{S}^1$ allows the transition of both ways to parameterize closed curves:

$$\begin{array}{rccc} \phi : & \mathbb{R} & \longrightarrow & \mathbb{S}^1 \\ & t & \longmapsto & (\cos t, \sin t). \end{array}$$

With that, a Jordan curve can be defined by a map $\alpha : [a, b] \to \mathbb{R}^2$ such that $\alpha(a) = \alpha(b)$ and the restriction of α to $[a, b[$ is injective. Notice that with the transition map ϕ, a $[0, 2\pi]$ domain is enough to describe the initial curve, but not to describe the movement of the chord when being about to complete a full lap. That is the counterpart of working with the injective domain of the initial curve, so the parameterization with $\mathbb{S}^1$ is more appropriate. As a remark, notice that we can also operate and differentiate functions defined in $\mathbb{S}^1$—seen as a differentiable manifold—by local calculus.

The study of the injectivity of the Holditch functions may be important. For closed curves defined on a closed interval I, one should identify $\inf I \sim \sup I$ to evaluate the injectivity or to take I to be half-closed from the beginning—for instance, $[0, 2\pi[$. Henceforth, the injectivity of the Holditch functions may be referred to as it has been just commented.

As a criterion, the movement of the chord will be assumed to follow, when possible (unless retrograde movements), the positive orientation of the curve.

As a final comment, recall that the Holditch functions depend on the chosen parameterization of α. The curve α will usually be parameterized by arc-length and the Holditch functions will be defined according to that parameterization.

3.1.2 Existence of Holditch curves

After having defined the Holditch functions, now the definition of Holditch curve follows directly.

Definition 3.4 (Holditch curve in the plane)**.** Let $\alpha : I \to \mathbb{R}^2$ be a counterclockwise oriented curve. Given $\ell > 0$ and $0 \leq \mathfrak{p} \leq \ell$, the *Holditch curve of α generated by a chord of length ℓ at a distance $\mathfrak{p}$ from the rear end* is defined by

$$H_\alpha(s) = \frac{1}{\ell}\Big((\ell - \mathfrak{p})\,\alpha\big(g(s)\big) + \mathfrak{p}\,\alpha\big(h(s)\big)\Big), \tag{3.4}$$

where g is the rear Holditch function and h the front Holditch function, i.e. they are continuous maps such that $\alpha\big(g(s)\big)$ and $\alpha\big(h(s)\big)$ describe the endpoints of the moving chord.

If $p \in [0, 1]$ is defined such that $\mathfrak{p} = p\,\ell$, then the *p-Holditch curve of α for the chord length ℓ* is defined as

$$H_\alpha(s) = (1 - p)\,\alpha\big(g(s)\big) + p\,\alpha\big(h(s)\big).$$

If $p = 1/2$, it is called the *midpoint Holditch curve*. By convenience and to shorten, if $\mathfrak{q} = \ell - \mathfrak{p}$, sometimes it is said that (3.4) is the Holditch curve generated by a chord of length $\ell = \mathfrak{p} + \mathfrak{q}$. Also, if there is no confusion and depending on the context, the lengths $\mathfrak{p}$ and $\mathfrak{q}$ will be denoted by p and q, respectively.

Of course, the *existence of Holditch curves* is reduced to the existence of the Holditch functions g and h.

Remark 3.5. If there is no retrograde motion for a chord length ℓ, the p-Holditch curve of α for such a length can be parameterized by

$$H_\alpha(s) = (1-p)\,\alpha(s) + p\,\alpha\big(f(s)\big), \tag{3.5}$$

where $f : I \to f(I)$ is the Holditch function for the parameterization α and the chord length ℓ.

See Figure 3.6 for an example of Holditch curve.

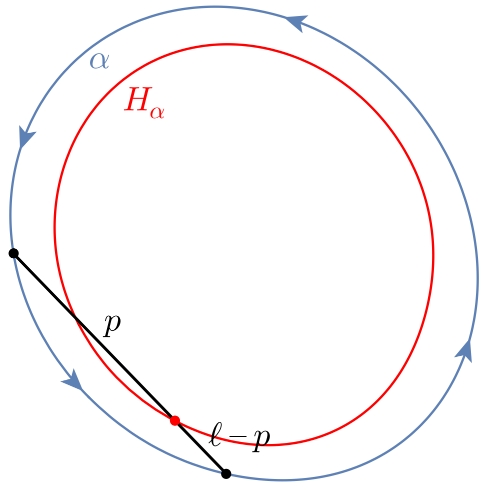

Figure 3.6: Example of a 0.76-Holditch curve for a chord length 2.

The following result is on the existence of Holditch curves generated by moving chords without retrograde motion. Its proof is a nice application of the *Implicit Function Theorem*. A similar result on the existence of Holditch curves can be found in [72], together with some of their smoothness and convexity properties.

Theorem 3.6. *Let $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ be a closed $\mathcal{C}^1$-curve. If $\ell > 0$ is such that*

(i) there is at least one possible position of a chord of length ℓ with end-points on the trace of α, and

(ii) for any $s \in \mathbb{S}^1$, the critical points of the distance function from $\alpha(s)$ restricted to the trace of α are at a distance not equal to ℓ,

then there exists a homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ as a Holditch function for α and the chord length ℓ. Therefore, there exist the p-Holditch curves of α for the chord length ℓ for any $p \in [0,1]$.

Proof. Define $F : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ as

$$F(s,t) = \big\|\alpha(t) - \alpha(s)\big\|^2.$$

The function F is $\mathcal{C}^1$ and

$$\frac{\partial F}{\partial t}(s,t) = 2\big\langle \alpha'(t),\, \alpha(t) - \alpha(s)\big\rangle.$$

Condition (i) ensures that there are $s_0, t_0 \in \mathbb{S}^1$ such that $F(s_0,t_0) = \ell^2$, i.e., an initial position for the chord is allowed. Moreover, condition (ii) is equivalent to saying that if $s, t \in \mathbb{S}^1$ are such that $\alpha(t) - \alpha(s)$ and $\alpha'(t)$ are orthogonal, then $\|\alpha(t) - \alpha(s)\| \neq \ell$. Since $\|\alpha(t_0) - \alpha(s_0)\| = \ell$, we have that $\big\langle \alpha'(t_0),\, \alpha(t_0) - \alpha(s_0)\big\rangle \neq 0$ (Figure 3.10 illustrates the geometric interpretation of the second condition).

Therefore, $\frac{\partial F}{\partial t}(s_0,t_0) \neq 0$ and the Implicit Function Theorem can be applied. There exist U_{s_0} neighborhood of s_0, V_{t_0} neighborhood of t_0 and a $\mathcal{C}^1$-function $f_0 : U_{s_0} \to V_{t_0}$ such that $f_0(s_0) = t_0$ and, for any $s \in U_{s_0}$,

$$F\big(s, f_0(s)\big) = \ell^2.$$

This means that for any $s \in U_{s_0}$, $\big\|\alpha(s) - \alpha\big(f_0(s)\big)\big\| = \ell$. We have shown that if there is one possible position of a chord of length ℓ with endpoints on the trace of α, specifically $\alpha(s_0)$ and $\alpha(t_0)$, then a piece of any p-Holditch curve can be built with the chord endpoints in neighborhoods of s_0 and t_0.

The extension of f_0, and therefore of the Holditch curve, to the whole $\mathbb{S}^1$ follows from a typical argument using the connection property of $\mathbb{S}^1$.

Let $A \subseteq \mathbb{S}^1$ be the biggest open subset where f_0 can be extended continuously as a Holditch function of α for the length ℓ. Denote by $f : A \to \mathbb{S}^1$ such an extension. Obviously, $A \neq \varnothing$ because $U_{s_0} \subseteq A$.

Now, define the function $\bar{F} : A \to \mathbb{R}$ as

$$\bar{F}(s) = F\big(s, f(s)\big).$$

Differentiating $\bar{F}$,

$$\bar{F}'(s) = \frac{\partial F}{\partial s}\big(s, f(s)\big) + \frac{\partial F}{\partial t}\big(s, f(s)\big)\, f'(s). \tag{3.6}$$

Since $\bar{F}$ is constant and equal to ℓ^2 by definition of f, then $\bar{F}'(s) = 0$ for all $s \in A$. Moreover, by condition (ii), for all $s, t \in A$,

$$\frac{\partial F}{\partial s}\big(s, f(s)\big) \neq 0 \quad \text{and} \quad \frac{\partial F}{\partial t}\big(s, f(s)\big) \neq 0.$$

Thus, from Equation (3.6), it is deduced that $f'(s) \neq 0$ for all $s \in A$. With that, we have proved that f is injective in A. Hence, $f : A \to f(A)$ is a homeomorphism defined on an open subset A so it can be extended continuously to its closure, which implies A to be closed.

Since $\mathbb{S}^1$ is connected, it is deduced that $A = \mathbb{S}^1$. This means that f_0 can be extended to the whole $\mathbb{S}^1$ as a bijective $\mathcal{C}^0$-function $f : \mathbb{S}^1 \to \mathbb{S}^1$. $\square$

Example 3.7. A *Cassini oval* with parameters a and b is the set of points such that the product of the distances from any point of the set to two fixed points $(-a, 0)$ and $(a, 0)$, called foci, is constant and equal to b^2.

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be the Cassini oval with foci $(-a, 0)$ and $(a, 0)$ and a constant product distance b^2 parameterized by

$$\alpha(t) = \rho(t)\,\big(\cos t,\, \sin t\big),$$

where

$$\rho(t) = \sqrt{a^2 \cos(2\,t) - \sqrt{b^4 - a^4 \sin^2(2\,t)}}.$$

Consider the case $a = \frac{19}{20}$ and $b = 1$, where the maximum orthogonal distance between two points in the curve is $\sqrt{39}/10 \approx 0.6245$. Take a moving chord of a bigger length, for instance, $\ell = 65/100$. Numerically it can be shown that the function

$$\big\|\alpha(t) - \alpha(s)\big\|^2 - \left(\frac{65}{100}\right)^2$$

attains a local minimum for $s_0 \approx 1.22$ and $t_0 \approx 4.86$, so that condition (ii) of Theorem 3.6 does not hold and therefore we cannot ensure the existence of a Holditch function. That can be due to retrograde movements or to the possibility of having a non-extensible Holditch function to the whole domain.

In this example, if the initial position of the chord is chosen according to the first point found at the length ℓ following the parameterization, then it can be shown that there exists a bijective Holditch function to the whole domain (see Figure 3.7). In such a case, any p-Holditch curve of α is a closed curve (see Figure 3.8-left). For other initial positions, the extension is not possible (a full lap cannot be completed, see Figure 3.8-right).

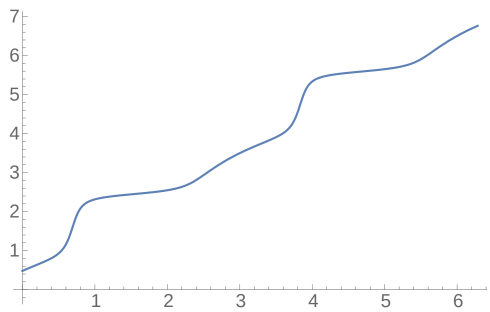

Figure 3.7: The Holditch function of α for $a = 19/20$ and $b = 1$ generated by a moving chord of length $\ell = 65/100$ is a well-defined homeomorphism to the whole domain for the natural initial position of the chord given by the parameterization.

Remark 3.8. Suppose α to be a closed curve. If the circle centered at any point of α and of radius ℓ has only two intersections with α, then retrograde motion won't appear and a bijective Holditch function $f : I \to f(I)$ can be easily defined by choosing one of these intersections continuously. The choice is usually done according to the positive orientation of α (see Figure 3.1).

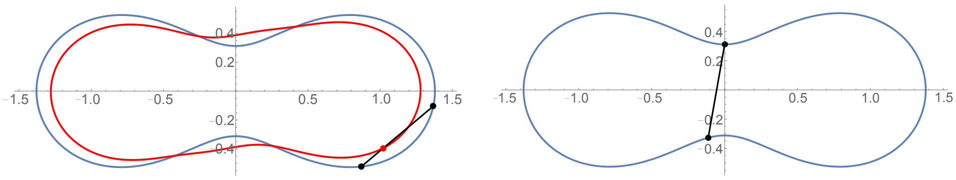

Figure 3.8: On the left, a p-Holditch curve of α for $a = 19/20$ and $b = 1$ generated by a moving chord of length $\ell = 65/100$. On the right, an example of an initial position for the same chord length for which the Holditch function can only be defined in a part of the curve.

Now, if at some point $\alpha(s_0)$ the intersection is reduced to just one point, $\alpha(t_0)$, then the circle is tangent to α at $\alpha(t_0)$, and thus

$$\big\langle \alpha'(t_0),\, \alpha(t_0) - \alpha(s_0) \big\rangle = 0.$$

In the situation of Example 3.7, Theorem 3.6 does not apply and an always forward movement for the chord cannot be ensured. Similarly, if the circle has three or more intersections with α, retrograde motion is not ensured, but it could appear. Notice that in Example 3.7 it is showed that having a chord length such that three or more intersections are found is not a sufficient condition for retrograde motion.

3.1.3 The important angles

In a Holditch motion there are some important angles which are useful to define and understand. Since the endpoints of the moving chord depend on the Holditch functions, the angles to be defined will also do.

First, recall the definition of an oriented angle in the plane.

Definition 3.9 (Oriented angle)**.** Given two vectors $\mathbf{u}$ and $\mathbf{v}$, the *oriented angle from* $\mathbf{u}$ *to* $\mathbf{v}$ is defined as the unique $\theta \in \,]-\pi, \pi]$ such that

$$\mathbf{v} = \cos\theta\, \mathbf{u} + \sin\theta\, J\mathbf{u}.$$

Nevertheless, since it is of interest to count the number of revolutions that a vector does in a motion, instead of having just an angle in $]-\pi, \pi]$, the notion of oriented angle function with codomain in $\mathbb{R}$ is needed (see [35]).

Definition 3.10 (Oriented angle function)**.** Let $\mathbf{u} : I \to \mathbb{R}^2$ be a vector field and $\big\{\mathbf{e}_1(s), \mathbf{e}_2(s)\big\}$ an orthonormal basis for each $s \in I$. Given $s_0 \in I$, choose $\theta_0 \in \mathbb{R}$ such that

$$\mathbf{u}(s_0) = \cos\theta_0\, \mathbf{e}_1(s_0) + \sin\theta_0\, \mathbf{e}_2(s_0).$$

Then there exists a unique differentiable function $\theta : I \to \mathbb{R}$ such that $\theta(s_0) = \theta_0$ and

$$\mathbf{u}(s) = \cos\theta(s)\, \mathbf{e}_1(s) + \sin\theta(s)\, \mathbf{e}_2(s).$$

The function θ is called the *oriented angle function from* $\mathbf{e}_1$ *to* $\mathbf{u}$ *determined by* θ_0.

When using the above definition, unless said otherwise, it will be assumed that $s_0 = \inf I \in I$ and θ_0 is the oriented angle from $\mathbf{e}_1(s_0)$ to $\mathbf{u}(s_0)$.

Let $\alpha : I \to \mathbb{R}^2$ and suppose that $\ell > 0$ is the length of the moving chord.

Define $\nu : J_\ell^g \to \mathbb{R}$ as the oriented angle function from $\mathbf{t}\big(g(s)\big)$ to $\alpha\big(h(s)\big) - \alpha\big(g(s)\big)$. Note that

$$\boxed{\Big\langle \mathbf{t}\big(g(s)\big),\, \alpha\big(h(s)\big) - \alpha\big(g(s)\big) \Big\rangle = \ell \cos \nu(s), \text{ for all } s \in J_\ell^g.} \tag{3.7}$$

Define $\mu : J_\ell^g \to \mathbb{R}$ as the oriented angle function from $\alpha\big(h(s)\big) - \alpha\big(g(s)\big)$ to $\mathbf{t}\big(h(s)\big)$. Thus,

$$\boxed{\Big\langle \mathbf{t}\big(h(s)\big),\, \alpha\big(h(s)\big) - \alpha\big(g(s)\big) \Big\rangle = \ell \cos \mu(s), \text{ for all } s \in J_\ell^g.} \tag{3.8}$$

Finally, define the angle between the tangents as $\phi(s) := \nu(s) + \mu(s)$, which is an oriented angle function from $\mathbf{t}\big(g(s)\big)$ to $\mathbf{t}\big(h(s)\big)$. Thus,

$$\boxed{\Big\langle \mathbf{t}\big(g(s)\big),\, \mathbf{t}\big(h(s)\big) \Big\rangle = \cos \phi(s), \text{ for all } s \in J_\ell^g.} \tag{3.9}$$

Notice that $\nu(s)$ may not necessarily correspond with the angle at $\alpha(s)$ because it is defined via the Holditch functions. It will only happen if there is no retrograde motion and $g = \mathrm{Id}$. A similar observation can be done with the other two angles. Moreover, since the Holditch functions depend on the chosen parameterization, these angles also do. For closed curves, recall that $J_\ell^g = I$. See Figure 3.9 to visualize the angles which have been defined above.

Other interesting angles can be defined in a Holditch setting. Let $\sigma : J_\ell^g \to \mathbb{R}$ be the oriented angle function from the positive OX axis to the tangent vector $\mathbf{t}\big(g(s)\big)$. Define $\theta := \nu + \sigma$, which is an oriented angle function from the positive OX axis to the Holditch chord placed at $\alpha\big(g(s)\big)$.

By the geometric interpretation of the curvature of a planar curve as the derivative of the turning angle, note that if α is arc-length parameterized, then

$$\theta'(s) = \nu'(s) + \kappa\big(g(s)\big),$$

where κ is the curvature of α. In Proposition 3.22 it will be shown that for strictly convex curves and chords without retrograde motion, the angle θ is monotone.

3.1.4 Characterization of retrograde motion

In this section, a sufficient condition on the length of the Holditch chord to avoid retrograde motion is given. Moreover, for strictly convex curves, a characterization of retrograde motion is found. The approach of [65] will be followed (see also the discussion in [70]).

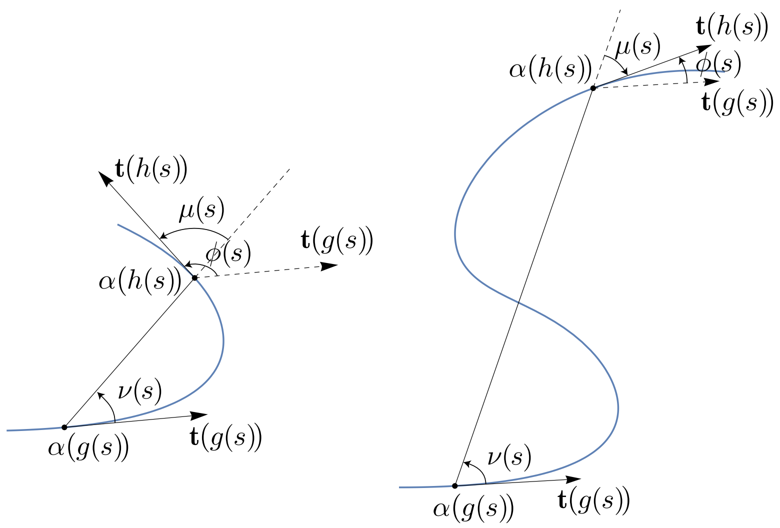

Figure 3.9: A determination of the oriented angle functions ν, μ and ϕ for a convex and a non-convex curve.

Definition 3.11 (Holditch radius)**.** Let $\alpha : I \to \mathbb{R}^2$ be a closed curve. Given $s \in I$, let $r(s)$ be the infimum radius of a circle centered at $\alpha(s)$ and tangent to the curve α at some point $\alpha(t)$. The *Holditch radius of* α is defined as

$$R_H := \inf_{s \in I} r(s).$$

In Figure 3.10 it is shown geometrically how the Holditch radius is defined.

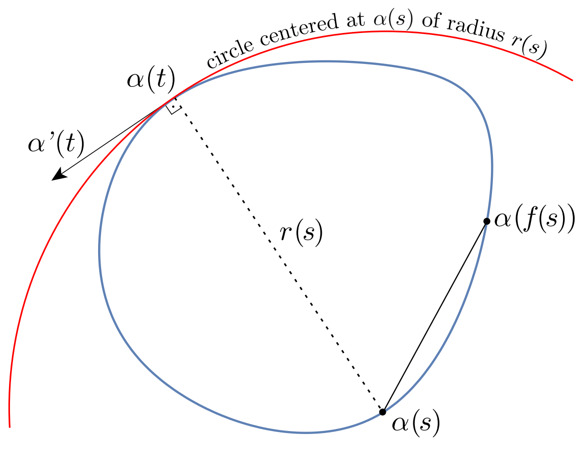

Figure 3.10: Definition of the Holditch radius as the infimum of the radii $r(s)$ for all $s \in I$.

Example 3.12. The Holditch radius of a circle is its diameter. If the curve is a square or an equilateral triangle, the Holditch radius is 0, but for a

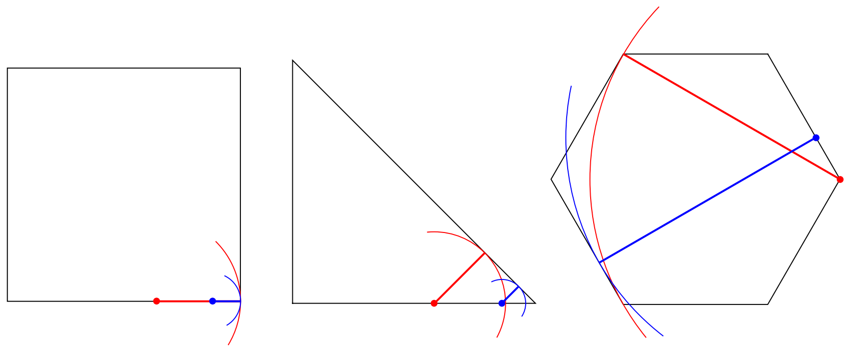

Figure 3.11: The Holditch radius of a square or an equilateral triangle is zero. In a regular hexagon it is the diameter of the radius of the inscribed circle (i.e., twice the apothem).

polygon with external angles at the vertices less than $\pi/2$, the Holditch radius is strictly positive (see Figure 3.11).

Example 3.13 (Computation of the Holditch radius for an ellipse)**.** Let $\alpha : [0, 2\pi] \to \mathbb{R}^2$ be the usual parameterization of an ellipse with semi-axes a and b:

$$\alpha(s) = \big(a\cos(s),\, b\sin(s)\big).$$

The computation of its Holditch radius is a typical minimization problem with restrictions. Let

$$F(s,t) = \big\|\alpha(t) - \alpha(s)\big\|^2.$$

We have to compute the non-zero minimum value of $F(s,t)$ under the restriction $\big\langle \alpha(t) - \alpha(s),\, \alpha'(t)\big\rangle = 0$. By symmetry, we can assume $t \in [\pi/2, \pi]$. Since

$$\begin{aligned}&\big\langle \alpha(t) - \alpha(s),\, \alpha'(t)\big\rangle\\ &\qquad = 2\,a^2\,\big(\cos(s) - \cos(t)\big)\sin(t) - 2\,b^2\cos(t)\,\big(\sin(s) - \sin(t)\big),\end{aligned}$$

we can solve $\langle \alpha(t) - \alpha(s),\, \alpha'(t)\rangle = 0$ in terms of t:

$$\begin{aligned} s = f(t) &:= -\mathbf{i}\,\log\left(-\frac{e^{-\mathbf{i}\,t}\big(b^2\cos(t) + \mathbf{i}\,a^2\sin(t)\big)}{b^2\,\cos(t) - \mathbf{i}\,a^2\sin(t)}\right)\\ &= \pi - t + \operatorname{atan2}\big(b^4\cos^2(t) - a^4\,\sin^2(t),\, a^2\,b^2\sin(2\,t)\big).\end{aligned}$$

Complex numbers are used just to simplify the expression; it is a real-valued function. Last equality with the 2-argument arctangent (see Definition 1.4) holds for $t \in [\pi/2,\, 3\,\pi/2]$.

By substituting it in $F(s,t)$ we get

$$G(t) := F\big(f(t), t\big) = \frac{2\,a^2\,b^2\,\big(a^2 + b^2 - (a^2 - b^2)\cos(2\,t)\big)^3}{\big(a^4 + b^4 - (a^4 - b^4)\cos(2\,t)\big)^2}.$$

The solutions of $G'(t) = 0$ are

$$0, \quad \pm\frac{\pi}{2}, \quad \pm\arccos\left(\pm\frac{a}{\sqrt{a^2-b^2}}\right) \quad \text{and} \quad \pm\arccos\left(\pm\frac{a\sqrt{a^2-2\,b^2}}{\sqrt{a^4-b^4}}\right),$$

and the corresponding values of $G(t)$ are

$$4\,a^2, \quad 4\,b^2, \quad 0, \quad \text{and} \quad \frac{27\,a^4\,b^4}{(a^2+b^2)^3}.$$

Excluding the 0, the minimum is the last one, $\frac{27\,a^4\,b^4}{(a^2+b^2)^3}$. Therefore, the Holditch radius for an ellipse of semi-axes a and b is

$$R_H = 3\sqrt{3}\,\frac{a^2\,b^2}{(a^2+b^2)^{3/2}}.$$

For example, taking $a = 2$ and $b = 1$, the Holditch radius is

$$\frac{12\sqrt{3}}{5\sqrt{5}} \approx 1.85903.$$

See in Figure 3.12 the computed Holditch radius.

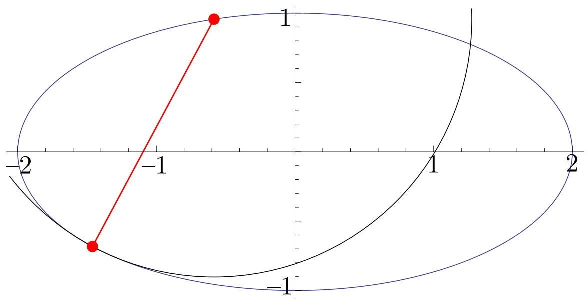

Figure 3.12: Visualization of the Holditch radius of an ellipse.

Theorem 3.14. *Let α be a simple closed planar $\mathcal{C}^1$-curve and let a Holditch chord of constant length $\ell > 0$ be placed somewhere with its endpoints on α. If $\ell < R_H$, then there is no retrograde motion for the Holditch chord.*

Proof. The condition $\ell < R_H$ implies condition (ii) of Theorem 3.6. Given an initial position of the chord—condition (i)—from that result there exists a homeomorphism as a Holditch function, so there is no retrograde motion. □

Example 3.15. The converse of Theorem 3.14 is false. Consider a constant width curve, for instance, $\alpha : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\alpha(s) = h(t)\,(\cos t,\, \sin t) + h'(t)\,(-\sin t,\, \cos t),$$

where the support function h is

$$h(t) = \frac{1}{12}\big(11 + \cos(3\,t)\big).$$

Consider the Holditch chord to be of length $\ell = 11$, i.e. its constant width (see Figure 3.13). In such a case, there is no retrograde motion when sliding the chord around the curve. Nevertheless, $\ell = R_H$.

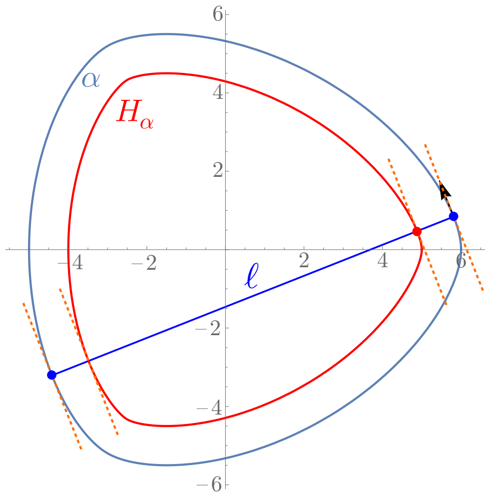

Figure 3.13: The curve α of constant width equal to the length ℓ of the Holditch chord. There is no retrograde motion but $\ell = R_H$. Its 1/11-Holditch curve is also a curve of constant width (see Proposition 3.28).

Example 3.16. Recall Example 3.7 of the Cassini oval α. Readily, the Holditch radius of α is $R_H = \sqrt{39}/10 \approx 0.6245$. It was shown that no retrograde motion appeared for a length $\ell = 0.65$ given a specific initial position, while for other initial positions a full turn was not possible. Bigger lengths without retrograde motion are allowed, see for example Figure 3.14, where a length $\ell = 0.8$ has been considered and retrograde motion does not appear.

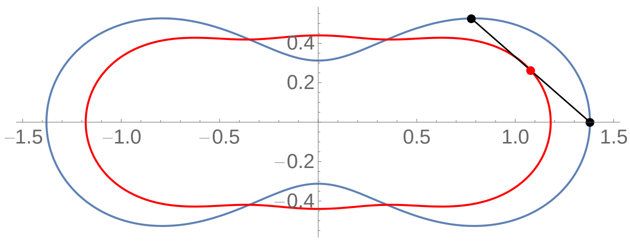

Figure 3.14: A Cassini oval and its 1/2-Holditch curve for a length $\ell = 0.8$ bigger than the Holditch radius.

Therefore, this example also shows that in general the converse of Theorem 3.14 is false and that the initial position of the chord is decisive.

However, although the converse of Theorem 3.14 is false in general, for strictly convex curves a characterization of retrograde motion can be given.

To have an orthogonal Holditch chord in one of the ends means that there exists some $s_0 \in I$ such that $\nu(s_0) = \frac{\pi}{2}$ or $\mu(s_0) = \frac{\pi}{2}$. Notice that

$\mu(s)$ can be interpreted as the angle which plays the role of $\nu(s)$ when the orientation of the curve is reversed to clockwise. If the chord moves following the orientation of the curve, then, by definition of the Holditch radius, we have that:

$$\boxed{\ell < R_H \text{ if and only if } \nu(s) < \frac{\pi}{2} \text{ and } \mu(s) < \frac{\pi}{2} \text{ for all } s \in I.} \tag{3.10}$$

Theorem 3.17. *Let α be a strictly convex simple closed and regular planar $\mathcal{C}^1$-curve and let $\ell > 0$ be a length less than any width of α. Suppose that a Holditch chord of length ℓ is placed somewhere with its endpoints on α. There is no retrograde motion for the Holditch chord if and only if $\ell < R_H$.*

Proof. That $\ell < R_H$ implies no retrograde motion on the chord is consequence of Theorem 3.14. Conversely, if there is no retrograde motion of the Holditch chord for a length $\ell > 0$ in α, then we have an injective function $f : I \to f(I)$ such that

$$\left\|\alpha(s) - \alpha\big(f(s)\big)\right\|^2 = \ell^2.$$

Differentiating this, we obtain

$$f'(s)\, \cos\mu(s) = \cos\nu(s).$$

Given $s \in I$, since $f'(s) > 0$, then $\cos\mu(s)$ and $\cos\nu(s)$ must have equal signs or are both zero. It means that given $s \in I$, $\nu(s), \mu(s) < \pi/2$, $\nu(s), \mu(s) > \pi/2$ or $\nu(s) = \mu(s) = \pi/2$.

If $s_0 \in I$ is such that $\nu(s_0) = \pi/2$, then $\mu(s_0) = \pi/2$, so ℓ is some width of α. Since ℓ is less than any width of α, then only two possibilities are allowed:

(i) for all $s \in I$, $\nu(s), \mu(s) < \pi/2$, or

(ii) for all $s \in I$, $\nu(s), \mu(s) > \pi/2$.

Without loss of generality, the case (i) can be assumed since case (ii) can be managed just by reversing the orientation of the curve (so that both angles would be less than $\pi/2$). Therefore, $\ell < R_H$. Notice that this discussion has been addressed to conclude (3.10) above. □

Example 3.18. In an *acute angle* formed by two rays, for any chord length $\ell > 0$, retrograde motion appears. Indeed, in that case, $R_H = 0$.

Consider now a rounded corner of an acute angle by considering a piece of a unit circle joining two lines, a horizontal one and one of slope -1 (see Figure 3.15). In this case, $R_H = 1 + \sqrt{2}$. Numerically it is seen that retrograde motion appears when exceeding the value R_H.

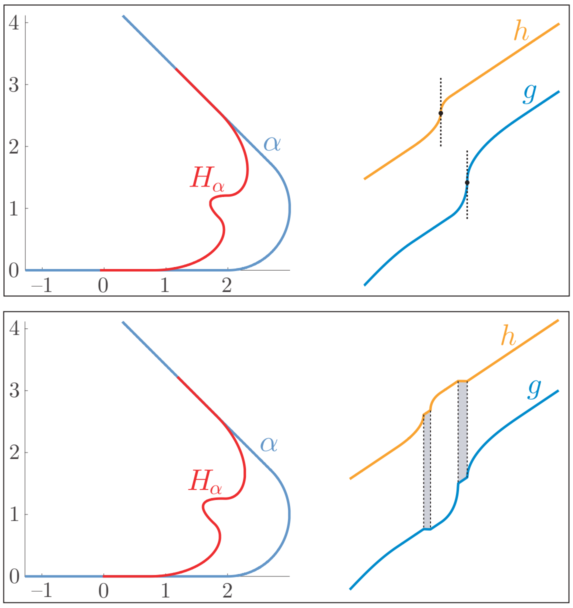

Figure 3.15: On top, the initial curve and its Holditch curve for a length $\ell = R_H = 1 + \sqrt{2}$, together with the plot of the Holditch functions g and h around the corner. Both functions are injective but at some points the tangent is vertical. On bottom, the same but for a length $\ell = 1.1 + \sqrt{2}$. Here, regions of retrograde movement appear in a neighborhood (shaded intervals) of where vertical tangents happened for $\ell = R_H$.

Remark 3.19. The study of the chord movement is more difficult if retrograde motion happens since different travels are allowed. That is because the circle centered at any point $\alpha(s)$ of the initial curve and of radius the chord length ℓ has more than two intersections with α. So, when going in or going out from a retrograde motion region there are different possibilities for the traveling of the chord and some criterion should be set.

Notice that the definition of the Holditch radius and Theorems 3.6 and 3.14 are also valid for non-convex curves. Therefore, to have $\ell < R_H$ and allowed an initial position for the chord is a sufficient condition to avoid retrograde motion for any curve. See in Figure 3.16 an example for a non-convex curve.

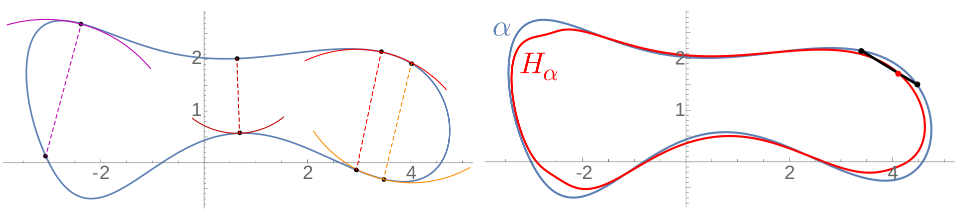

Figure 3.16: On the left, a few tangent circles to the initial curve α centered at points of it. On the right, the 1/3-Holditch curve of α for a chord length $\ell = 1.25$, which is less than the Holditch radius (and therefore, less than any radius of those circles). No retrograde motion appears for this chord length.

3.2 Some results in the Holditch setting

In this section, a few results on the Holditch setting are given, regarding the angle function θ, the regularity of Holditch curves, Holditch curves of constant width curves and Barbier's theorem.

3.2.1 The Holditch angle function

Here it is shown that for a strictly convex curve without retrograde motion, the angle function which describe the moving chord (seen as an indicatrix) is strictly increasing. In this part, if not stated otherwise, the following hypothesis will be assumed.

Hypothesis $\mathcal{H}(\alpha, \ell, f)$**.** Suppose a Holditch motion of a moving chord of length $\ell > 0$ without retrograde motion along a strictly convex closed regular curve $\alpha : I \to \mathbb{R}^2$ parameterized by arc length and positively oriented. Associated with this parameterization and the chord length ℓ, define the Holditch function f such that

$$\Big\|\alpha\big(f(s)\big) - \alpha(s)\Big\| = \ell,$$

for all $s \in I$.

Recall that the definition of the angles ν, μ, ϕ, σ and θ of Section 3.1.3 is given according to the Holditch functions. Henceforth, under the assumption $\mathcal{H}(\alpha, \ell, f)$, these angles will be defined according to the Holditch function f (see Figure 3.17 as a reminder of their definitions).

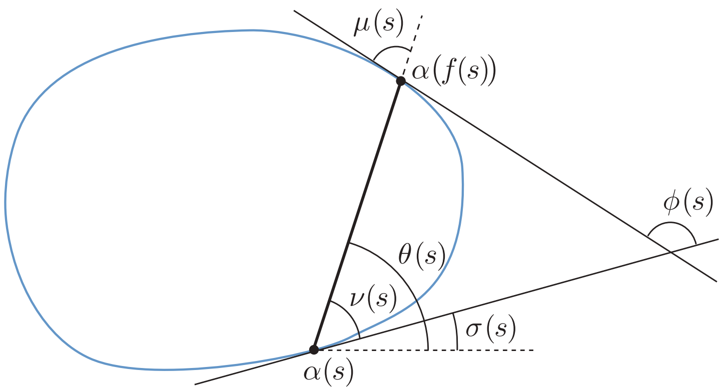

Figure 3.17: The important angles for a non-retrograde motion of the chord along a convex curve.

Recall that from the definition of the angle θ, we have that $\theta'(s) = \nu'(s) + \kappa(s)$ for all $s \in I$.

Lemma 3.20. *If the hypothesis $\mathcal{H}(\alpha, \ell, f)$ holds, then*

$$f'(s) \cos \mu(s) = \cos \nu(s).$$

Proof. The Holditch constraint is to have

$$\left\|\alpha\big(f(s)\big) - \alpha(s)\right\|^2 = \ell^2$$

for all $s \in I$. Differentiating this expression, we get

$$f'(s)\left\langle \mathbf{t}\big(f(s)\big),\, \alpha\big(f(s)\big) - \alpha(s)\right\rangle = \left\langle \mathbf{t}(s),\, \alpha\big(f(s)\big) - \alpha(s)\right\rangle,$$

which can be rewritten as in the statement. $\square$

Lemma 3.21. *If the hypothesis $\mathcal{H}(\alpha, \ell, f)$ holds, then*

$$1 - f'(s)\cos\phi(s) = \ell\left(\nu'(s) + \kappa(s)\right)\sin\nu(s)$$

for all $s \in I$.

Proof. From the definition of ν, we have Equation (3.7):

$$\left\langle \mathbf{t}(s),\, \alpha\big(f(s)\big) - \alpha(s)\right\rangle = \ell\cos\nu(s).$$

Differentiating this expression:

$$\kappa(s)\left\langle \mathbf{n}(s),\, \alpha\big(f(s)\big) - \alpha(s)\right\rangle + f'(s)\cos\phi(s) - 1 = -\ell\,\nu'(s)\sin\nu(s).$$

Since

$$\left\langle \mathbf{n}(s),\, \alpha\big(f(s)\big) - \alpha(s)\right\rangle = \ell\,\sin\nu(s),$$

the result is obtained just by arranging the terms. $\square$

As a consequence, the following result can be deduced.[1]

Proposition 3.22. *If α is a strictly convex closed regular curve and there is no retrograde motion for a chord length $\ell > 0$, then the angle function θ is strictly monotone.*

Proof. Without loss of generality, suppose α to be positively oriented and arc-length parameterized and let ν, θ and ϕ be positive. By convexity, $\theta(s) > \nu(s) > 0$ and $\phi(s) > \nu(s)$ for all $s \in I$. From Lemma 3.21, we have that

$$\theta'(s) = \nu'(s) + \kappa(s) = \frac{1 - f'(s)\cos\phi(s)}{\ell\sin\nu(s)},$$

where $\sin\nu(s) > 0$. Now, from Lemma 3.20, since $\mu(s) = \phi(s) - \nu(s)$,

$$f'(s) = \frac{\cos\nu(s)}{\cos\big(\phi(s) - \nu(s)\big)},$$

so we have that

$$1 - f'(s)\cos\phi(s) = 1 - \frac{\cos\phi(s)\cos\nu(s)}{\cos\big(\phi(s) - \nu(s)\big)} = \frac{\sin\phi(s)\sin\nu(s)}{\cos\big(\phi(s) - \nu(s)\big)} > 0.$$

This implies $\theta'(s) > 0$ for all $s \in I$, so that θ is strictly increasing. $\square$

[1] The proof of Proposition 3.22 came from interesting discussions on Holditch's theorem I had during my stay in Denmark with my supervisor Steen Markvorsen.

Proposition 3.23. *Under the hypothesis $\mathcal{H}(\alpha, \ell, f)$,*

$$\phi'(s) = \kappa\big(f(s)\big)\, f'(s) - \kappa(s)$$

for any $s \in I$.

Proof. Notice that

$$\phi(s) = \sigma\big(f(s)\big) - \sigma(s). \tag{3.11}$$

Since $\sigma'(s) = \kappa(s)$, the result is obtained by differentiating (3.11). □

Immediately, from Proposition 3.23, the following corollary is deduced by integration.

Corollary 3.24. *Under the hypothesis $\mathcal{H}(\alpha, \ell, f)$,*

$$\phi(s) = \int_s^{f(s)} \kappa(t)\, \mathrm{d}t$$

for any $s \in I$.

3.2.2 Singularities of Holditch curves

The next theorem states a sufficient condition on the chord length to ensure regularity in the Holditch curves of a regular curve.

Theorem 3.25. *Let α be a regular simple closed planar $\mathcal{C}^1$-curve and let a chord of length $\ell > 0$ be placed somewhere with its endpoints on α. If ℓ is less than the Holditch radius of α, then any p-Holditch curve is regular (and the moving chord has no retrograde movements).*

Proof. By Theorem 3.14, no retrograde movements appear in the moving chord. Given $p \in [0, \ell]$, the p/ℓ-Holditch curve of α can be parameterized as

$$H_\alpha(s) = \alpha(s) + p\,\big(\cos\nu(s)\,\mathbf{t}(s) + \sin\nu(s)\,\mathbf{n}(s)\big).$$

We have that

$$\begin{aligned} H'_\alpha(s) = \Big(1 - p\,\sin\nu(s)\,\big(\nu'(s) + \kappa(s)\big)\Big)\,\mathbf{t}(s) \\ + p\,\cos\nu(s)\,\big(\nu'(s) + \kappa(s)\big)\,\mathbf{n}(s). \end{aligned} \tag{3.12}$$

Thus,

$$\begin{aligned} \big\|H'_\alpha(s)\big\|^2 &= 1 - 2\,p\,\sin\nu(s)\,\big(\nu'(s) + \kappa(s)\big) + p^2\,\big(\nu'(s) + \kappa(s)\big)^2 \\ &= \cos^2\nu(s) + \Big(\sin\nu(s) - p\,\big(\nu'(s) + \kappa(s)\big)\Big)^2. \end{aligned} \tag{3.13}$$

By definition of the Holditch radius, we have that $\nu(s) \neq \pm\pi/2$ for all $s \in I$. Therefore, (3.13) is never zero and the regularity of H_α is ensured for any $p \in [0, \ell]$. □

Remark 3.26. Similar results to these given here related to the regularity of Holditch curves can be found in [72]. There it is proved that given a convex regular curve α, if the circle of radius ℓ and centered at any point of α cuts the curve at precisely two points and if in addition the chord is never perpendicular to the initial curve at both endpoints at the same time, then the regularity of its Holditch curves is ensured.

3.2.3 Holditch curves of constant width curves

In this section, given a constant width curve, the properties of their Holditch curves are studied. First, observe that any moving chord on a curve of constant width has no retrograde movements if the chord length is less than its width.

Theorem 3.27. *Let α be a regular $\mathcal{C}^1$-curve of constant width d. Any chord of length $0 < \ell < d$ placed somewhere on α moves without retrograde motion.*

Proof. Readily, the Holditch radius of α is its width d. Thus, since ℓ is less than the Holditch radius of α, by Theorem 3.14 the result follows. □

From now on, consider the case of a Holditch chord of length equal to the constant width.

Theorem 3.28. *Let α be a positively oriented regular curve of constant width ℓ. Any p/ℓ-Holditch curve of α for the chord length ℓ is a (inner) parallel curve to α at a distance p and it is itself a curve of constant width $\ell - 2\,p$.*

Proof. Suppose $\alpha : [0, 2\pi] \to \mathbb{R}^2$ is parameterized by a support function (and extended to $\mathbb{R}$ by periodicity). In this setting, we have that $\nu(s) = \pi/2$ for all $s \in I$ (see Section 3.1.4). Therefore, given $p \in [0, \ell]$, the p/ℓ-Holditch curve of α for the chord length ℓ takes the form:

$$H_\alpha(s) = \alpha(s) + p\,\mathbf{n}(s),$$

which is the expression of the inner parallel curve to α at a distance p (see Section 1.2).

The condition of α being of constant width ℓ can be written as

$$\big\|\alpha(s+\pi) - \alpha(s)\big\| = \ell.$$

Since $\mathbf{n}(s+\pi) = -\mathbf{n}(s)$ and $\big\langle \alpha(s+\pi) - \alpha(s), \mathbf{n}(s)\big\rangle = \ell$, we have that

$$\begin{aligned}\Big\|H_\alpha(s+\pi) - H_\alpha(s)\Big\|^2 &= \big\|\alpha(s+\pi) - \alpha(s) - 2\,p\,\mathbf{n}(s)\big\|^2 \\ &= \ell^2 + 4\,p^2 - 4\,p\,\ell = (\ell - 2\,p)^2.\end{aligned}$$

Therefore H_α is of constant width equal to $\ell - 2\,p$. □

Remark 3.29. We have actually proved that the inner parallel curve to a constant width curve is of constant width. It can also be seen for outer offsets: the outer parallel curve at a distance p to a curve of constant width ℓ is also of constant width equal to $\ell + 2\,p$.

Proposition 3.30. *Let α be a positively oriented regular curve of constant width ℓ. If*

$$p < \frac{1}{\kappa_{sup}},$$

where $\kappa_{sup} = \sup_{s\in I}\big|\kappa(s)\big|$, then the p/ℓ-Holditch curve of α for the chord length ℓ is regular.

Proof. By Theorem 3.28, the p/ℓ-Holditch curve of α for a chord length ℓ is a inner parallel curve to α at a distance p. It suffices now to use Proposition 1.6 to have the statement. □

3.2.4 Barbier's theorem as a limiting case

Barbier's theorem (Theorem 1.19) states that every curve of constant width has perimeter π times its width. An elementary proof of this result can be derived from Holditch's theorem and Steiner's formula for parallel curves. In fact, Barbier's theorem can be seen as an infinitesimal version of Holditch's when taking the length of the Holditch chord equal to the width ℓ of the curve.

Proof of Barbier's theorem via Holditch's theorem. Let L be the perimeter of the constant width curve. If we take the length of the Holditch chord equal to ℓ, then for $0 < p < \ell$, the p/ℓ-Holditch curve of the constant width curve is an inner offset to the initial curve at a distance p (Theorem 3.28). Take p sufficiently small to have a regular and simple offset (see Proposition 3.30).

On the one hand, by Steiner's formulae (Theorem 1.8, page 8), the area between both curves is equal to $L\,p - \pi\,p^2$. On the other hand, by Holditch's theorem, that area is equal to $\pi\,p\,(\ell - p)$. Therefore, from both expressions we deduce that $L = \pi\,\ell$. □

3.3 The Holditch map

Given $\ell > 0$, denote by $\mathcal{C}^1_{\ell-\mathrm{fwd}}$ (resp. $\mathcal{P}^1_{\ell-\mathrm{fwd}}$) the space of simple closed $\mathcal{C}^1$ (resp. $\mathcal{C}^1$-piecewise) curves such that there are no retrograde movements in a Holditch motion for a chord length ℓ (only forward movements are allowed).

Definition 3.31 (Holditch map)**.** Given $\ell > 0$ and $p \in [0,1]$, the *smooth Holditch map* is defined as the map $\mathcal{H}_{\ell,p} : \mathcal{C}^1_{\ell-\mathrm{fwd}}(\mathbb{S}^1, \mathbb{R}^2) \to \mathcal{C}^0(\mathbb{S}^1, \mathbb{R}^2)$ which sends any simple closed $\mathcal{C}^1$-curve $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ into its p-Holditch curve for a chord length ℓ.

Analogously, the *piecewise-smooth Holditch map* is defined as the map $\tilde{\mathcal{H}}_{\ell,p} : \mathcal{P}^1_{\ell-\mathrm{fwd}}(\mathbb{S}^1, \mathbb{R}^2) \to \mathcal{C}^0(\mathbb{S}^1, \mathbb{R}^2)$ that sends any simple closed $\mathcal{C}^1$-piecewise curve $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ into its p-Holditch curve for a chord length ℓ.

If there is no ambiguity in the considered domain, any of the defined maps above will be called the *Holditch map*.

In the next lemma it is proved that given $\ell > 0$, for near simple closed curves without retrograde movements for a chord length ℓ, near Holditch functions are defined by such a length. It is a preliminary result to show later that the Holditch map is uniformly continuous (Theorem 3.33).

Lemma 3.32. *Let $\ell > 0$ and $\alpha, \beta \in \mathcal{C}^1_{\ell-\mathrm{fwd}}$. Suppose that $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ are the Holditch functions for α and β (respectively) and a chord length ℓ. Given $\eta > 0$, there exists $\delta > 0$ such that if $\big\|\alpha(s) - \beta(s)\big\| < \delta$ for all $s \in \mathbb{S}^1$, then $\big\|f(s) - g(s)\big\| < \eta$ for all $s \in \mathbb{S}^1$.*

Proof. Given two differentiable maps, $h_1, h_2 : \mathbb{S}^1 \to \mathbb{R}^2$, consider

$$F : (\mathbb{S}^1 \times \mathbb{R}) \times (\mathbb{S}^1 \times \mathbb{R}) \to \mathbb{R}^2$$

defined by

$$F(s,u,t,v) = \left(\Big\| \alpha(t) + v\, h_2(t) - \big(\alpha(s) + u\, h_1(s)\big) \Big\|^2,\ u - v \right).$$

Since $\alpha \in \mathcal{C}^1_{\ell-\mathrm{fwd}}$, then for all $s_0 \in \mathbb{S}^1$ there is $t_0 = f(s_0)$ such that $\big\|\alpha(t_0) - \alpha(s_0)\big\| = \ell$. Equivalently, $F(s_0, 0, t_0, 0) = (\ell^2, 0)$. Now, it is easy to check that the 2×2 matrix given by the partial derivatives

$$\begin{aligned} \left.\frac{\partial F}{\partial t}\right|_{(s_0,0,t_0,0)} &= \Big(2\,\big\langle \alpha'(t_0),\, \alpha(t_0) - \alpha(s_0)\big\rangle,\ 0\Big), \\ \left.\frac{\partial F}{\partial v}\right|_{(s_0,0,t_0,0)} &= (*,\ -1), \end{aligned}$$

has maximal rank. Thus, the Implicit Function Theorem can be applied: there are neighborhoods U_{s_0} of $(s_0, 0)$ and V_{t_0} of $(t_0, 0)$ and a continuous map $f^{s_0} : U_{s_0} \to V_{t_0}$ such that $f^{s_0}(s_0, 0) = (t_0, 0)$ and, for all $(s,u) \in U_{s_0}$,

$$F\big(s, u, f^{s_0}(s,u)\big) = (\ell^2, 0). \tag{3.14}$$

If we write $f^{s_0}(s,u) = \big(f_1^{s_0}(s,u),\, f_2^{s_0}(s,u)\big)$, then (3.14) is equivalent to

$$\begin{aligned} f_2^{s_0}(s,u) &= u, \\ \Big\|\alpha\big(f_1^{s_0}(s,u)\big) + f_2^{s_0}(s,u)\, h_2\big(f_1^{s_0}(s,u)\big) - \big(\alpha(s) + u\, h_1(s)\big)\Big\| &= \ell. \end{aligned} \tag{3.15}$$

With the first equality, the second one can be written as follows:

$$\Big\|\alpha\big(f_1^{s_0}(s,u)\big) + u\, h_2\big(f_1^{s_0}(s,u)\big) - \big(\alpha(s) + u\, h_1(s)\big)\Big\| = \ell. \tag{3.16}$$

By a similar argument as in the proof of Theorem 3.6, the function f^{s_0} can be extended to a homeomorphism $\tilde{f} : \mathbb{S}^1 \times U \to \mathbb{S}^1 \times U$ with $U = \,]{-\delta_0},\, \delta_0[$ and $\delta_0 > 0$. Let $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$. Since $\tilde{f}_1 : \mathbb{S}^1 \times U \to \mathbb{S}^1$ is uniformly continuous, given $\eta > 0$, there is $\delta > 0$, which can be supposed to be less than δ_0, such that if $|u| < \delta$ for any $u \in U$, then

$$\big\|\tilde{f}_1(s,u) - \tilde{f}_1(s,0)\big\| < \eta$$

for all $s \in \mathbb{S}^1$ and $u \in U$. Therefore, if $\|\alpha(s) - \beta(s)\| < \delta$ for all $s \in \mathbb{S}^1$, then

$$\Big\|\tilde{f}_1\Big(s, \big\|\alpha(s) - \beta(s)\big\|\Big) - \tilde{f}_1(s,0)\Big\| < \eta$$

for all $s \in \mathbb{S}^1$. Now, it suffices to prove that

$$\tilde{f}_1(s,0) = f(s) \quad \text{and} \quad \tilde{f}_1\Big(s, \big\|\alpha(s) - \beta(s)\big\|\Big) = g(s). \tag{3.17}$$

Notice that we can write

$$\beta(s) = \alpha(s) + \|\alpha(s) - \beta(s)\|\, h_1(s)$$

where

$$h_1(s) := \frac{\beta(s) - \alpha(s)}{\big\|\beta(s) - \alpha(s)\big\|}.$$

Also,

$$\beta\big(\tilde{f}_1(s,u)\big) = \alpha\big(\tilde{f}_1(s,u)\big) + \|\alpha(s) - \beta(s)\|\, h_2\big(\tilde{f}_1(s,u)\big),$$

where

$$h_2(t) := \frac{\beta(t) - \alpha(t)}{\Big\|\beta\big((\pi_1 \circ \tilde{f}_1^{-1})(t)\big) - \alpha\big((\pi_1 \circ \tilde{f}_1^{-1})(t)\big)\Big\|},$$

with π_1 being the projection onto the first coordinate. Suppose that h_1 and h_2 are extended by continuity where they are not defined. With that, from (3.15) and (3.16) used for the extension $\tilde{f}$—and evaluated in $u = 0$ and $u = \big\|\alpha(s) - \beta(s)\big\|$—we deduce (3.17), as it was desired. □

Theorem 3.33. *The smooth Holditch map is uniformly continuous.*

Proof. Let $\ell > 0$ and $p \in [0,1]$. We must show that given $\epsilon > 0$, there exists $\delta > 0$ such that if $\alpha, \beta \in \mathcal{C}^1_{\ell-\mathrm{fwd}}(\mathbb{S}^1, \mathbb{R}^2)$ are such that $\|\alpha(s) - \beta(s)\| < \delta$ for all $s \in \mathbb{S}^1$, then

$$\big\|H_\alpha(s) - H_\beta(s)\big\| < \epsilon$$

for all $s \in \mathbb{S}^1$.

First, notice that the curve $\beta : \mathbb{S}^1 \to \mathbb{R}^2$ is a continuous map defined on a compact set, so it is uniformly continuous; that is, given $\frac{\epsilon}{2}$, there is $\eta > 0$ such that for all $s, t \in \mathbb{S}^1$, $\|s - t\| < \eta$, we have $\|\beta(s) - \beta(t)\| < \frac{\epsilon}{2}$.

Now, thanks to Lemma 3.32, given η, there is $\delta > 0$, which can be supposed to be less than $\frac{\epsilon}{2}$, such that if $\big\|\alpha(s) - \beta(s)\big\| < \delta$ for all $s \in \mathbb{S}^1$, then $\big\|f(s) - g(s)\big\| < \eta$ for all $s \in \mathbb{S}^1$. Therefore, by being β uniformly continuous,

$$\Big\|\beta\big(f(s)\big) - \beta\big(g(s)\big)\Big\| < \frac{\epsilon}{2}. \tag{3.18}$$

Suppose that $\|\alpha(s) - \beta(s)\| < \delta$ for all $s \in \mathbb{S}^1$. Thus, by using (3.18),

$$\begin{aligned} \Big\|\alpha\big(f(s)\big) - \beta\big(g(s)\big)\Big\| &= \Big\|\alpha\big(f(s)\big) - \beta\big(f(s)\big) + \beta\big(f(s)\big) - \beta\big(g(s)\big)\Big\| \\ &\le \Big\|\alpha\big(f(s)\big) - \beta\big(f(s)\big)\Big\| + \Big\|\beta\big(f(s)\big) - \beta\big(g(s)\big)\Big\| \\ &\le \delta + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{3.19}$$

Therefore, if $\|\alpha(s) - \beta(s)\| < \delta$ for all $s \in \mathbb{S}^1$, as a consequence of the inequality (3.19) we can conclude that

$$\begin{aligned} \big\|H_\alpha(s) - H_\beta(s)\big\| &\le (1-p)\,\big\|\alpha(s) - \beta(s)\big\| + p\,\Big\|\alpha\big(f(s)\big) - \beta\big(g(s)\big)\Big\| \\ &< (1-p)\,\epsilon + p\,\epsilon = \epsilon. \end{aligned}$$

∎

Corollary 3.34. *The piecewise-smooth Holditch map is uniformly continuous.*

Proof. Let $\alpha, \beta \in \mathcal{P}^1_{\ell-\mathrm{fwd}}$ and $\epsilon > 0$. By the Stone–Weierstrass theorem (see [77], page 159), there exist sequences of $\mathcal{C}^\infty$-curves $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \beta.$$

In particular, there exists some $k \in \mathbb{N}$ such that if $n \ge k$, both α_n and β_n have no retrograde movements for the chord length ℓ. Therefore, for any $n \ge k$, $\alpha_n, \beta_n \in \mathcal{C}^1_{\ell-\mathrm{fwd}}$. By Theorem 3.33, for any $n \ge k$, given $\epsilon > 0$ there is $\delta > 0$ such that if $\big\|\alpha_n(s) - \beta_n(s)\big\| < \delta/3$ for all $s \in \mathbb{S}^1$, then

$$\big\|H_{\alpha_n}(s) - H_{\beta_n}(s)\big\| < \frac{\epsilon}{3} \tag{3.20}$$

for all $s \in \mathbb{S}^1$.

Moreover, there is $n_0 \in \mathbb{N}$, which can be supposed to be $n_0 \geq k$, such that if $n \geq n_0$,

$$\left\|\alpha(s) - \alpha_n(s)\right\| < \frac{\delta}{3} \quad \text{and} \quad \left\|\beta(s) - \beta_n(s)\right\| < \frac{\delta}{3},$$

for all $s \in \mathbb{S}^1$. Therefore,

$$\left\|\alpha(s) - \beta(s)\right\| \leq \left\|\alpha(s) - \alpha_n(s)\right\| + \left\|\alpha_n(s) - \beta_n(s)\right\| + \left\|\beta_n(s) - \beta(s)\right\| < \delta$$

for all $s \in \mathbb{S}^1$.

Also, by the continuity of the Holditch map for the sequences $(\alpha_n)_{n=n_0}^{\infty}$ and $(\beta_n)_{n=n_0}^{\infty}$ (Theorem 3.33), given $\epsilon > 0$, we have that there exists $n_1 \geq n_0$ such that if $n \geq n_1$, then

$$\left\|H_\alpha(s) - H_{\alpha_n}(s)\right\| < \frac{\epsilon}{3} \quad \text{and} \quad \left\|H_\beta(s) - H_{\beta_n}(s)\right\| < \frac{\epsilon}{3}. \tag{3.21}$$

Thus, by using (3.20) and (3.21), given $\epsilon > 0$, we have found $\delta > 0$ such that if $\left\|\alpha(s) - \beta(s)\right\| < \delta$ for all $s \in \mathbb{S}^1$, then

$$\begin{aligned}\left\|H_\alpha(s) - H_\beta(s)\right\| \leq{}& \left\|H_\alpha(s) - H_{\alpha_n}(s)\right\| \\ &+ \left\|H_{\alpha_n}(s) - H_{\beta_n}(s)\right\| + \left\|H_{\beta_n}(s) - H_\beta(s)\right\| < \epsilon\end{aligned}$$

for all $s \in \mathbb{S}^1$ (and where it suffices to take any $n \geq n_1$). □

The continuity of the Holditch map will be used in next section to ensure the convergence of a sequence of Holditch curves.

3.4 Unveiling the Holditch ellipse

One of the most intriguing facts of the classical Holditch statement is that the area between the initial curve and its Holditch curve constructed by a chord length $\ell = p + q$ is equal to the area $\pi\, p\, q$ of an ellipse with semi-axes p and q. Nevertheless, no ellipses appear in the statement of the theorem nor in its proof, so many authors have asked for any geometrical interpretation of that mysterious ellipse in the Holditch setting.

The approach followed in this section is the one given in [65].[2] The idea is to deduce a polygonal version of Holditch's theorem where the involved ellipse is explicitly seen. Later, by polygonal approximations, the smooth case will be addressed. To do this properly, the continuity of the Holditch map defined and studied in Section 3.3 will be used.

To come up with a polygonal version of Holditch's theorem, basic and easy cases of Holditch curves are considered first.

3.4.1 Holditch curve associated with an angle

An approach to the treatment of the initial curve as a polygon corresponds to considering two rays which emanate from one point and that subtend a definite angle. We will refer to the *external angle* in a corner formed by two rays as the oriented angle from the first side to the second according to the orientation of the polygonal curve (see Figure 3.18).

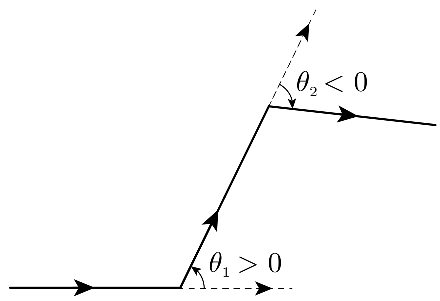

Figure 3.18: The external angle in a corner of a polygonal curve is the oriented angle from the first side to the second following the orientation of the curve.

Suppose now that the moving chord is shorter than the lengths of all the edges of the polygon. When both chord endpoints move along the same edge, the corresponding piece of Holditch curve is a straight line segment which coincides with part of that edge, so there is no gap there between both curves. The interesting case comes when each chord endpoint is on a different side of the polygon. That is the situation to be studied in this section.

[2]Thanks to Mark Cooker, whose comments were useful to improve the cited paper and, as a consequence, this text.

The fact that the resulting curve is a piece of an ellipse has been stated before by many authors, such as [13], [36] or [96]. The contribution here is to relate the ellipse corresponding to an arbitrary angle with the ellipse corresponding to a right angle through a shear transformation.

The Holditch ellipse will be called *oblique* if its semi-axes are not parallel to any of the two rays, otherwise, it will be called *orthogonal*.

First, recall the definition of a shear transformation.

Definition 3.35 (Shear transformation)**.** A *horizontal shear transformation* is a linear map $S_c : \mathbb{R}^2 \to \mathbb{R}^2$, with $c \in \mathbb{R}$, defined by

$$S_c(x, y) = (x + c\, y,\, y).$$

If, in addition, a horizontal translation of vector x_0 is applied, the resulting map will be denoted by $S_{c,x_0} : \mathbb{R}^2 \to \mathbb{R}^2$ and its expression is

$$S_{c,x_0}(x, y) = (x + x_0 + c\, y,\, y).$$

There are two main properties of shear mappings that are needed. They are stated in the next proposition.

Proposition 3.36. *Let S_c be a horizontal shear transformation with $c \neq 0$. Then*

(i) S_c is area-preserving, and

(ii) S_c transforms orthogonal ellipses into oblique ellipses.

Proof. On the one hand, since the Jacobian of S_c is

$$\begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1,$$

the map S_c is area-preserving.

On the other hand, if an orthogonal ellipse with semi-axes a and b is considered,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then its image by the shear S_c has equation

$$\frac{(x + c\, y)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This can be rewritten as

$$A\, x^2 + B\, x\, y + C\, y^2 + F = 0,$$

where

$$A = \frac{1}{a^2}, \quad B = \frac{2\,c}{a^2}, \quad C = \frac{1}{b^2} + \frac{c^2}{a^2} \quad \text{and} \quad F = -1.$$

Since the discriminant of this non-degenerate conic is

$$B^2 - 4\,A\,C = \frac{4}{a^2\,b^2} < 0,$$

then it is the equation of an ellipse (but oblique—it has the term xy). □

Remark 3.37. As usual, if an oblique ellipse

$$A\,x^2 + B\,xy + C\,y^2 + F = 0$$

is obtained by shearing an orthogonal one and we want to know its new semi-axes, then it suffices to make a rotation of angle

$$-\frac{1}{2}\arctan\left(\frac{B}{A - C}\right)$$

to put its axes parallel to the Cartesian axes and to find the new equation.

Remark 3.38 (On the area of an elliptic sector)**.** Analogously as in Definition 3.35, a *vertical shear transformation* is defined as a linear map $\bar{S}_c : \mathbb{R}^2 \to \mathbb{R}^2$, with $d \in \mathbb{R}$, defined by

$$\bar{S}_d(x, y) = (x,\, y + d\,x).$$

and it verifies the same properties seen in Proposition 3.36: it is area preserving and it transforms ellipses into ellipses.

With that, we have that a parameterization of the kind

$$(a, d)\cos(t) + (c, b)\sin(t), \quad t \in [0, 2\pi],$$

is an ellipse (it is an orthogonal ellipse with semi-axes a and b sheared with maps S_c and $\bar{S}_d$). Recall that the parameter t does not represent the oriented angle from the positive OX axis to the position vector of the ellipse as happens in the circle. Nevertheless, an easy formula for the area of an elliptic sector can be given in terms of the parameter t, that is, a formula for the elliptic sector described when the parameter t moves along the values from t_1 to t_2. Such an elliptic sector will be called a (t_1, t_2)-*elliptic sector.*

The region to compute its area can be described by the planar surface $\gamma :]0, 1[\times]t_1, t_2[\to \mathbb{R}^2 \times \{0\}$ given by

$$\gamma(\rho, t) = \rho\,\big((a, d, 0)\cos(t) + (c, b, 0)\sin(t)\big).$$

It is easy to show that

$$\big\|(\gamma_\rho \wedge \gamma_t)(\rho, t)\big\| = \rho\,|a\,b - c\,d|.$$

Therefore, the area of a (t_1, t_2)-elliptic sector is equal to

$$\int_{t_1}^{t_2} \int_0^1 \rho\, |a\, b - c\, d|\; \mathrm{d}\rho\; \mathrm{d}t = \frac{1}{2}\, |a\, b - c\, d|\, (t_2 - t_1). \tag{3.22}$$

Therefore, the area of (t_1, t_2)-sector of an orthogonal ellipse with semi-axes a and b is

$$\frac{1}{2}\, a\, b\, (t_2 - t_1),$$

and the same result holds if only a horizontal shear is done ($c \neq 0$, $d = 0$) or just a vertical one ($c = 0$, $d \neq 0$).

The following result indicates the shape of the Holditch curve in the case we are focusing on and shows a way to find out where the area of the Holditch ellipse comes from. It may be appropriate to remark, as stated in [96]—page 65—that the Holditch curve associated with an angle is a particular case of the ellipse construction studied by Leonardo da Vinci (the interested reader can look over the da Vinci's ellipsograph or the trammel of Archimedes, [17]).

Proposition 3.39. *The Holditch curve defined by two rays forming an external angle θ and a chord length $\ell = p + q$ is a piece of an oblique ellipse (see Figure 3.19) obtained by a shear transformation of an orthogonal one with semi-axes p and q and with the same area: $\frac{1}{2}\, p\, q\, \theta$.*

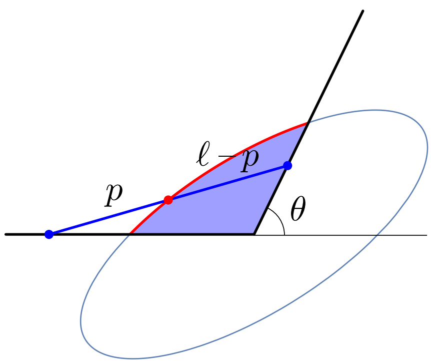

Figure 3.19: The Holditch curve defined by two rays for a chord length $\ell = p + q$ is an arc of an oblique ellipse which in turn is the image by a shear of an orthogonal ellipse with semi-axes p and q. Its area is $\frac{1}{2}\, p\, q\, \theta$, where θ is the external angle.

Proof. Denote by θ the external angle formed by the two rays and let t be the angle between the Holditch chord and the first ray. Without loss of generality, suppose $\theta > 0$ (the other case can be managed just by changing the orientation of both rays or by symmetry). The quantity t will be the parameter of the Holditch curve. The situation is represented in Figure 3.20.

Set the origin of the coordinates, $O = (0, 0)$, at the intersection of both rays. Let $A = A(t)$ and $B = B(t)$ be the points defined by the ends of the

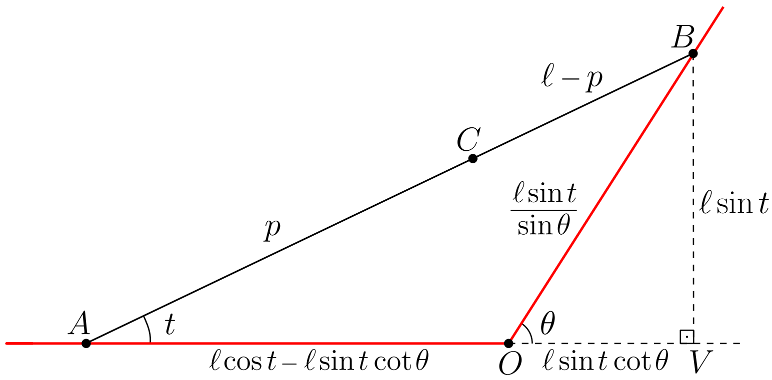

Figure 3.20: Parameterization of the p/ℓ-Holditch curve of two rays for a chord length ℓ.

chord placed in these rays. The only thing which is needed to parameterize the Holditch curve is to know the coordinates of A and B, since such a parameterization can be defined as

$$\gamma(t) = \frac{1}{\ell}\big((\ell - p)\,A + p\,B\big), \quad t \in [0, \theta]. \tag{3.23}$$

If we focus on the triangle OVB and call its hypotenuse v, then we have $\ell \sin(t) = v \sin(\theta)$, from which it follows that $v = \ell\,\frac{\sin(t)}{\sin(\theta)}$. Thus, the side OV of the triangle has length $v \cos(\theta) = \ell \sin(t) \cot(\theta)$. From this, we get

$$B = \big(\ell \sin(t) \cot(\theta),\ \ell \sin(t)\big).$$

Now, we focus on the triangle AVB. If the length of the side AO is called x, then $\ell \cos(t) = x + \ell \sin(t) \cot(\theta)$. From that, we obtain $x = \ell \cos(t) - \ell \sin(t) \cot(\theta)$. Thus, we deduce that

$$A = \big(-\ell \cos(t) + \ell \sin(t) \cot(\theta),\ 0\big).$$

By replacing the points A and B in (3.23), we conclude that

$$\gamma_{\theta, p, \ell}(t) = \big(-(\ell - p)\cos(t) + \ell \sin(t) \cot(\theta),\ p \sin(t)\big), \quad t \in [0, \theta], \tag{3.24}$$

is the parameterization of the p/ℓ-Holditch curve.

In particular, notice that when $\theta = \pi/2$, the expression (3.24) reduces to

$$\gamma_{\frac{\pi}{2}, p, \ell}(t) = \big(-(\ell - p)\cos(t),\ p \sin(t)\big), \quad t \in [0, \pi/2],$$

which is the parameterization of an ellipse with semi-axes lengths p and $\ell - p$.

Finally, it is easy to check that it is the image of a horizontal shear transformation:

$$\gamma_{\theta, p, \ell} = S_{\frac{\ell}{p}\cot(\theta)}\Big(\gamma_{\frac{\pi}{2}, p, \ell}(t)\Big), \quad t \in [0, \theta].$$

This means that the Holditch curve defined by two rays with external angle θ is the image by a shear of a $(0, \theta)$-arc of an orthogonal ellipse with semi-axes lengths p and $\ell - p$. Remember that shears transform orthogonal ellipses into oblique ellipses —Proposition 3.36(ii). Then, the Holditch curve is an arc of an oblique ellipse.

Furthermore, since shear transformations preserve areas—Proposition 3.36(i)— the area defined by the two rays and the Holditch curve is equal to the area of a sector of an orthogonal ellipse with semi-axes lengths p and $\ell - p$. $\square$

Once the explicit parameterization of the Holditch curve defined by a chord sliding on two rays forming a corner is known, Equation (3.24), some easy examples can be given.

Example 3.40. If α is a square or an equilateral triangle, the Holditch ellipse can be explicitly seen. Figure 3.21 shows a Holditch curve in these examples. In the square, four parts of the Holditch ellipse with semi-axes p and $\ell - p$ appear (in the case $\ell = 2p$, they are four parts of a circle). In the equilateral triangle, three pieces of an oblique ellipse (obtained by a shear transformation of an orthogonal one with semi-axes p and $\ell - p$, as stated in Proposition 3.39) are found. From the fact that shear maps preserve areas, it is deduced that the area in both examples (fixed ℓ and p) is the same and equal to the area of the corresponding Holditch ellipse.

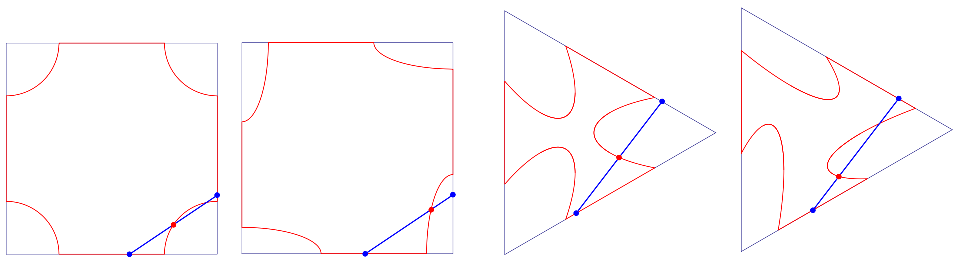

Figure 3.21: Holditch curves in a square and in an equilateral triangle for different choices of p.

The basic result given in Proposition 3.39 will be used in the following two sections, in which a separate study is done depending on the length of the chord.

3.4.2 Polygons with long sides

According to the previous section, it is now easy to deal with polygons with all sides longer than the length of the moving chord.

Proposition 3.41. *Let α be a closed polygonal curve. Let H_α be the Holditch curve of α generated by a chord length $\ell = p + q$. If the sides of α are of length bigger than or equal to ℓ, then*

$$\mathcal{A}(\alpha) - \mathcal{A}(H_\alpha) = \pi\, p\, q.$$

Proof. Suppose that the initial curve α is an n-sided closed polygon with external angles θ_i, for $i = 1, \dots n$, defined at each vertex. By Proposition 3.39, the area at each vertex between the polygon and the Holditch curve is the area of a $(0, \theta_i)$-sector of an orthogonal ellipse with semi-axes lengths p and q. That area is well known and equal to

$$\frac{\theta_i}{2}\, p\, q.$$

Therefore, if we have n vertices, the total Holditch area is

$$\sum_{i=1}^{n} \frac{\theta_i}{2}\, p\, q = \frac{1}{2}\, p\, q \sum_{i=1}^{n} \theta_i.$$

Now, $\sum_{i=1}^{n} \theta_i = 2\,\pi$ since it is the sum of all the external angles in a simple closed polygon. Thus, the result of the statement is found. Notice that the Holditch area in vertices with negative external angles count as negative areas when making the difference of areas (see in Figure 3.22 an example of a non-convex closed polygon). □

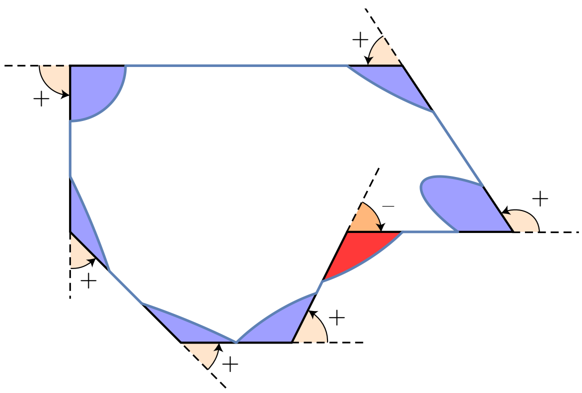

Figure 3.22: Example of a non-convex closed polygonal curve and its midpoint Holditch curve for a chord length equal to its minor side. The total external angle is $2\,\pi$. For negative external angles, negative areas are computed, so that the total Holditch area remains as $\pi\, p\, q$.

3.4.3 Cutting off a corner

It has just been seen how to deal with the case of a polygonal curve with all sides longer than the length of the moving chord (Proposition 3.41). Now any polygonal curve will be considered. So the situation to be managed is when the endpoints of the moving chord are not on consecutive sides of the polygon. In such a case, the Holditch curve will be piecewise defined as it is shown in Figure 3.23.

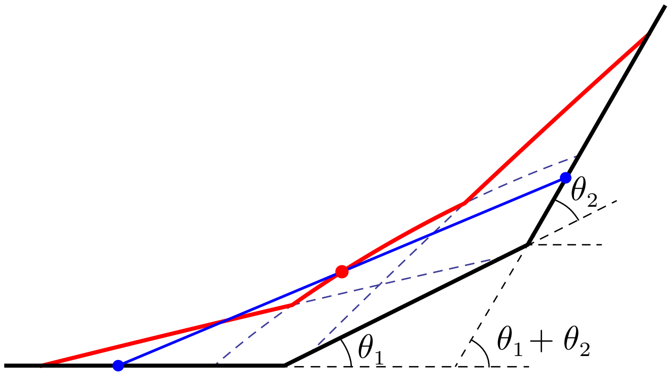

Figure 3.23: A triangular region of area is cut off at the corner of angle $\theta_1 + \theta_2$.

The idea to solve this general setting is to reduce it to the known studied case. The fact is to think that the only thing we have done to the case of two rays is to cut off an old corner with a straight line, as shown in Figure 3.23. The process of cutting off a corner in that setting will be referred to as a *cutting-off corner step*. The different pieces of the Holditch curve can be parameterized separately as done in Proposition 3.39 taking care of which external angle is in play. That is because each piece of the curve is a translation and rotation of the same parameterization (3.24), taking the appropriate angle. The change points can be obtained with a short calculation in each case.

Proposition 3.42. *The amount of Holditch area defined by two rays forming a corner is preserved if such a corner is cut off by a segment joining both rays. Such an area is equal to* $\frac{1}{2}\,p\,q\,\theta$*, where* θ *is the external angle formed by both rays and* $\ell = p + q$ *is the chord length.*

Proof. By Proposition 3.41, the area of the elliptic sector defined by a chord of constant length $\ell = p + q$ traveling on two rays forming an external angle θ is equal to

$$\frac{1}{2}\,p\,q\,\theta. \tag{3.25}$$

If the corner is cut off as in Figure 3.24, then two pieces of new elliptic sectors appear while the old elliptic sector is of a smaller angle and it is cut off. The sum of the three full elliptic sector areas is

$$\frac{1}{2}\,p\,q\,\phi_1 + \frac{1}{2}\,p\,q\,(\phi_2 - \phi_1) + \frac{1}{2}\,p\,q\,(\theta - \phi_2), \tag{3.26}$$

where ϕ_1 and ϕ_2 are the angles that form the first ray with the Holditch chord when changing from one elliptic sector to another. These angles will be referred to as the *changing angles*.

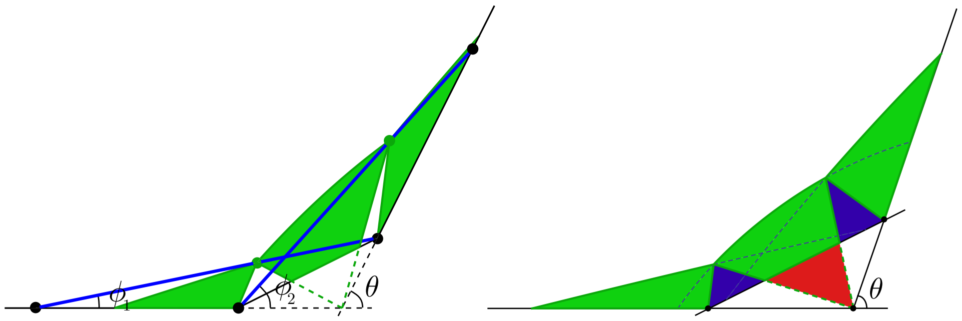

Figure 3.24: New elliptic sectors appear when cutting off a corner (left). The old elliptic sector (the middle one) is smaller and the cut off area by the middle side is the same as the area of the triangles that appear when considering the Holditch area (right).

Since (3.26) is equal to (3.25), there is no change of area when cutting off a corner. It means that the cut off area of the middle elliptic sector is the same as what is needed to get the Holditch area (see Figure 3.24-right). □

A different proof for the area conservation was given in [65] by means of shear transformations. The reader can find this procedure in the next section. Now, the general theorem for any convex polygonal curve can be stated.

Theorem 3.43. *Let α be a polygonal convex curve. If H_α is the Holditch curve of α generated by a chord length $\ell = p + q$, then*

$$\mathcal{A}(\alpha) - \mathcal{A}(H_\alpha) = \frac{1}{2}\, p\, q\, \theta_{\text{total}},$$

where θ_{total} is the total amount of swept external angle. Moreover, if α is closed,

$$\mathcal{A}(\alpha) - \mathcal{A}(H_\alpha) = \pi\, p\, q.$$

Proof. It follows directly from Proposition 3.42, which can be extended if more than two corners are involved. Moreover, if the polygon is a closed curve, then $\theta_{\text{total}} = 2\pi$ and the classical result is recovered but for any convex closed polygon: $\pi\, p\, q$. □

3.4.4 Area preserving by shear transformations

In the process of cutting off corners, the Holditch curve in each piece is an oblique ellipse obtained by a shear defined by some angle. Therefore, it is of interest to study the intersection between two oblique ellipses obtained by shearing an orthogonal one.

Lemma 3.44. *Let $\mathcal{C}_1$ be the oblique ellipse defined as the image of an orthogonal ellipse by the horizontal shear transformation, $S_{c_1}(x, y) = (x + c_1\, y,\, y)$,* *and let $\mathcal{C}_2$ be the oblique ellipse defined as the image of the same orthogonal ellipse by another horizontal shear transformation plus a horizontal translation, $S_{c_2,x_0}(x, y) = (x + x_0 + c_2\, y,\ y)$ with $x_0 < 0$ (see Figure 3.25). Suppose that $c_1 < c_2$. Let A_i be the leftmost intersection between the line joining the centers O_1 and O_2 of both ellipses and the ellipse $\mathcal{C}_i$, $i = 1, 2$. Among the four possible intersection points between both ellipses, let P be the one located above that line and to the left. The area defined by the three points A_1, A_2 and P is equal to the area of a triangle with a base length $-x_0$ and height $\frac{x_0}{c_1 - c_2}$.*

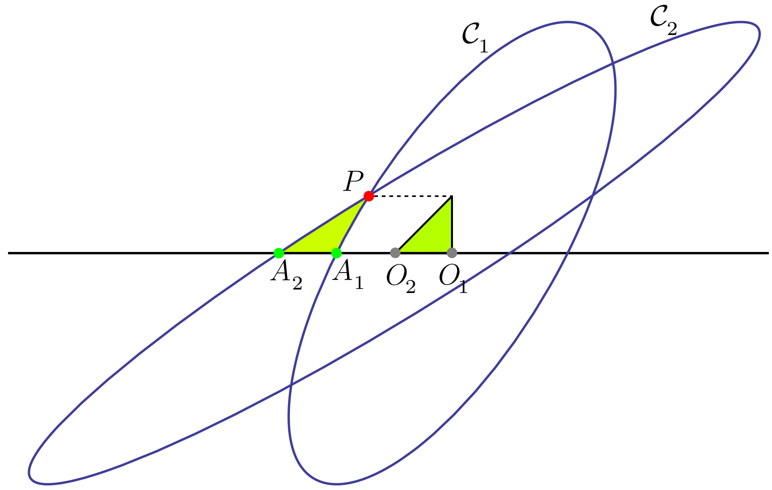

Figure 3.25: The area of both regions is the same.

Proof. If we apply $S_{c_1-c_2}$ to the second ellipse, then what we get is the same first ellipse $\mathcal{C}_1$ but translated according to the vector $(x_0, 0)$. The region (P, A_2, C)—see Figure 3.26—is transformed into the region defined by the points (Q, A_2, C). The area we are looking for is the area of (P, A_2, C) minus the area of (P, A_1, C). Since shear transformations preserve areas, the

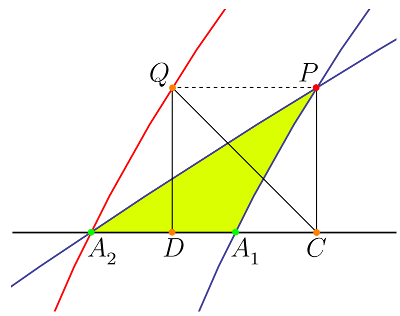

Figure 3.26: Enlargement of the significative area.

area of (P, A_2, C) is the same as the area of (Q, A_2, C). Moreover, the area of (P, A_1, C) is the same as the area of (Q, A_2, D). Therefore, the area of (P, A_2, A_1) is the same as the area of the triangle (Q, D, C). The length of its base is the norm of the vector $(x_0, 0)$ and its height is the height of the intersection point P. A straightforward computation shows that the second coordinate of P is $\frac{x_0}{c_1 - c_2}$. $\square$

The following idea illustrates the purpose of the previous result. The fact is to prove that the area of the triangle removed by the new line when cutting off a corner is equal to the area of the two new bumps of Holditch area which appear on each side (see Figure 3.27).

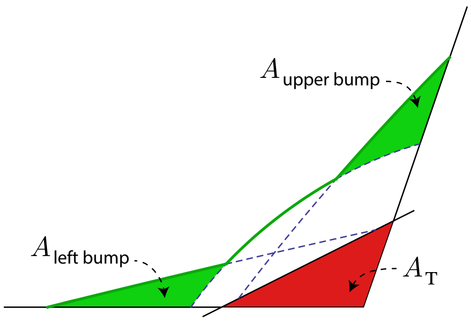

Figure 3.27: The area of the removed triangle is equal to the sum of the areas defined by the two new pieces of the Holditch curve.

Denote by $A_{\mathbf{T}}$ the area of the triangle defined by the cutting straight line and the initial corner and by $A_{\text{left bump}}$ and $A_{\text{upper bump}}$ the two areas defined by the Holditch curve before the cutting-off corner step and the new Holditch curve for the cut-off corner.

Proposition 3.45. *Given $p \in [0, 1]$, in a cutting-off corner step for the p-Holditch curve of two rays forming a corner,*

$$A_{\text{left bump}} = p\, A_{\mathbf{T}} \quad \textit{and} \qquad A_{\text{upper bump}} = (1-p)\, A_{\mathbf{T}}.$$

Proof. Suppose now that the chord length is $\ell = p + q$, with $p \in [0, \ell]$. We will apply Lemma 3.44 to the situation shown in Figure 3.28. In this case, both shears are defined by the parameters (according to the notation in the statement of Lemma 3.44):

$$c_1 = \frac{\ell}{p} \cot(\theta_1 + \theta_2), \quad c_2 = \frac{\ell}{p} \cot(\theta_1) \quad \text{and} \quad x_0,$$

where $c_1 < c_2$ and $x_0 < 0$. Therefore, the area of the left bump is

$$\begin{aligned} A_{\text{left bump}} &= \frac{1}{2} \frac{x_0^2}{c_2 - c_1} = \frac{1}{2} \frac{p}{\ell} \frac{x_0^2}{\cot(\theta_1) - \cot(\theta_1 + \theta_2)} \\ &= \frac{p}{2\,\ell} \frac{x_0^2 \sin(\theta_1 + \theta_2) \sin(\theta_1)}{\sin(\theta_2)}. \end{aligned}$$

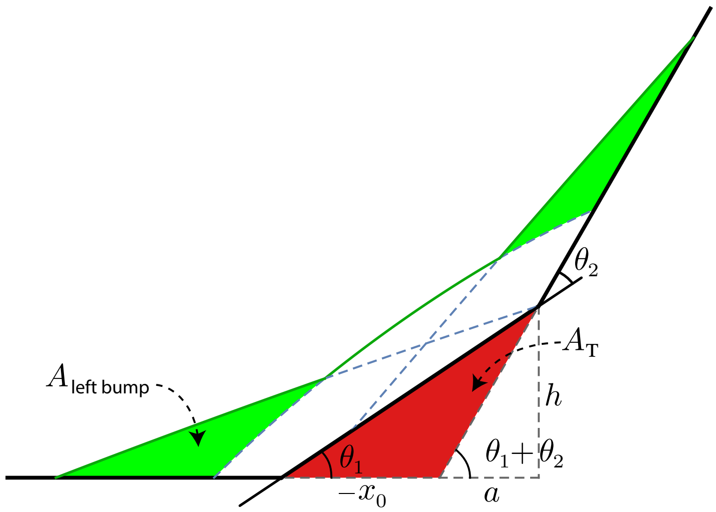

Figure 3.28: Two rays forming a corner of external angle $\theta_1+\theta_2$ cut off by a straight line. Two new corners of external angles θ_1 and θ_2 with each ray, respectively, appear.

On the other hand, let's compute the area of the triangle. We can get its height from the system of equations

$$\begin{cases} \frac{h}{-x_0+a} & = \tan(\theta_1), \\ \frac{h}{a} & = \tan(\theta_1+\theta_2), \end{cases}$$

where $a \in \mathbb{R}$. The solution is

$$h = -\frac{x_0\, \sin(\theta_1+\theta_2)\sin(\theta_1)}{\sin(\theta_2)}.$$

Hence, the area of the triangle is

$$A_{\mathbf{T}} = -\frac{x_0\, h}{2} = \frac{1}{2}\,\frac{x_0^2\, \sin(\theta_1+\theta_2)\sin(\theta_1)}{\sin(\theta_2)} = \frac{\ell}{p}\, A_{\text{left bump}},$$

which proves that the area of the left bump is equal to p/ℓ times the area of the triangle.

Now, thanks to a flip with respect to the bisecting line of the inner angle at the corner, we can interchange both bumps (see Figure 3.29). Notice that the flip also interchanges the lengths p and $\ell - p$ in the chord. Hence, the area of the second bump is equal to $(\ell-p)/\ell$ times the area of the removed triangle. □

Theorem 3.46. *The Holditch area defined by two rays forming a corner is preserved by a cutting-off corner step.*

Proof. If the p-Holditch curve of two rays forming a corner for a chord length ℓ does not cut the straight line of a cutting-off corner step (Figure 3.27

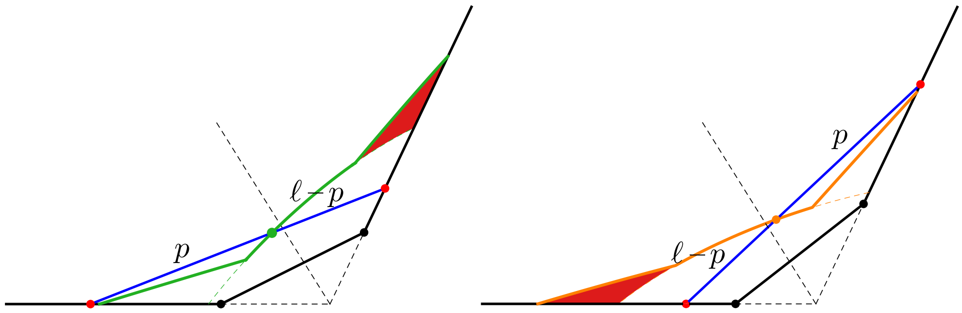

Figure 3.29: Representation of the flip.

or 3.28), it is clear that the statement holds by Proposition 3.45: $A_{\mathbf{T}} = A_{\text{left bump}} + A_{\text{upper bump}}$. Otherwise, the removed area in a cutting-off corner step, A_{removed}, is not the full area inside the triangle (see Figure 3.30).

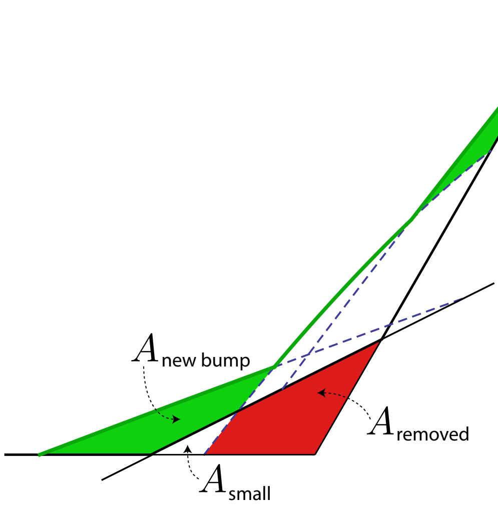

Figure 3.30: Case when the straight line which cuts off the corner also cuts the old Holditch curve.

Suppose, without loss of generality, that the left bump is the one with problems (as in Figure 3.30). By Proposition 3.45,

$$A_{\text{left bump}} + A_{\text{upper bump}} = A_{\mathbf{T}}.$$

Following the notation of Figure 3.30, since $A_{\mathbf{T}} = A_{\text{removed}} + A_{\text{small}}$ and $A_{\text{left bump}} = A_{\text{new bump}} + A_{\text{small}}$, so we conclude that

$$A_{\text{new bump}} + A_{\text{upper bump}} = A_{\text{removed}},$$

which proves the area preservation.

If the problem appears in both bumps, we can argue similarly with each of them to reach the same conclusion, i.e., the sum of the areas of both bumps is equal to the removed area. $\square$

3.4.5 "Cut-and-paste" construction of the Holditch ellipse

Given a rectifiable convex closed curve $\alpha : \mathbb{S}^1 \to \mathbb{R}^2$ with non-vanishing curvature and a natural number $n > 1$, build a convex polygon, P_n, with 2^n sides as follows. For any $i \in \{0, 1, 2, \ldots, 2^n - 1\}$, let L_i be the tangent line to the curve α at $\alpha\left(\frac{2\pi\, i}{2^n}\right)$. The vertices of the polygon are the intersection points between two consecutive tangent lines, L_i and $L_{i+1 \pmod{2^n}}$, and the sides are the tangent segments between two vertices.

Each polygon, P_n, has an associated Holditch curve, H_n. At each term, the area between H_n and P_n is equal to the area of an ellipse with semi-axes p and $\ell - p$. The process of passing from one term in the sequence $\{P_n\}_{n=2}^{\infty}$ to the next one consists in a finite number of cutting-off corner steps, as can be seen in Figure 3.31.

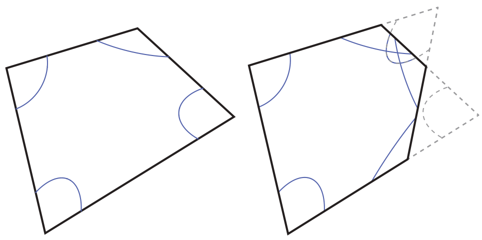

Figure 3.31: A corner cut off by the construction of the sequence of polygons.

Figure 3.32 shows the three first terms of the sequence of Holditch curves associated with a circle and also with a convex curve. Both examples clearly show the fast convergence of the sequences—it takes a small number of terms to attain a good approximation.

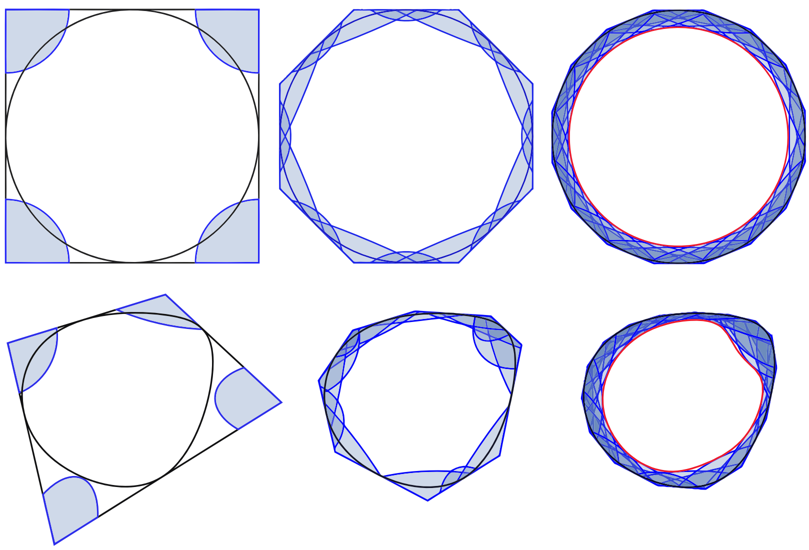

Figure 3.32: The three first terms in the sequence towards the Holditch curve of a circle (above) and of a convex planar curve (below).

Since the sequence of polygons $\{P_n\}_{n=2}^{\infty}$ is convergent to the curve α and the Holditch map is (uniformly) continuous (Corollary 3.34), then the sequence of Holditch curves $\{H_n\}_{n=2}^{\infty}$ is convergent to the Holditch curve of α. Intuitively, this shows that the Holditch area in the smooth case is the result of an infinite number of cutting-off corner steps which preserve and distribute the area of the Holditch ellipse by pasting small pieces of it.

Remark 3.47. If the initial curve α is rectifiable (and not necessarily closed), Holditch's theorem can be deduced as a corollary of Theorem 3.43 thanks to polygonal approximations and area preservation as before. In the limiting case, the Holditch area takes the form

$$\mathcal{A}(\alpha) - \mathcal{A}(H_\alpha) = \frac{1}{2}\, p\, q \int_I \theta'(s)\, \mathrm{d}s = \frac{1}{2}\, p\, q \int_I \kappa(s)\, \mathrm{d}s.$$

For closed curves, $\int_I \kappa(s)\, \mathrm{d}s = 2\pi$ and hence Holditch's theorem is recovered. The interested reader can see the discussion of [70].

3.5 Generated curves in the plane

The name of *generated curves* of the title will refer to curves generated as a sum of an initial curve α and a vector field along α. The Holditch construction reminds to other similar constructions involving chords of constant length, such as the motion of a bicycle, constant width curves or parallel curves. The general kind of curves under the name of *generated curves* are defined in such a way that all these constructions can be seen as particular cases. The results on areas and lengths reviewed in Chapter 1 for these curves will be given now as a corollary of a more general result for generated curves (Lemma 3.50). In particular, another proof of Holditch's theorem will be given together with its extension for two initial curves in the plane. As a main reference for this section, the reader can see [79].

This section, although not entirely new, it may be interesting as an easy introduction in the plane for the analogous results which will be given in Section 5.3 for 2-dimensional constant curvature manifolds.

Let $\alpha : I \to \mathbb{R}^2$ be a regular simple closed curve parameterized by arc-length. For each point $\alpha(s)$, consider a unique unit direction $\beta(s)$ given by a function $\beta : I \to \mathbb{R}^2$. This function can be described by means of the angle function $\theta : I \to \mathbb{R}$ defined in Section 3.1.3 as

$$\beta(s) = \big(\cos\theta(s), \sin\theta(s)\big).$$

The angle $\theta(s)$ is an oriented angle function from a fixed direction, namely the positive OX axis, to the direction $\beta(s)$. Moreover, recall that $\nu : I \to \mathbb{R}$ is the oriented angle function from $\mathbf{t}(s)$ to the unit direction $\beta(s)$. Thus,

$$\beta(s) = \cos\nu(s)\,\mathbf{t}(s) + \sin\nu(s)\,\mathbf{n}(s).$$

We will assume that β can be extended continuously by periodicity.

Definition 3.48 (Generated curve in the plane)**.** Under the setting above, given a continuous function $p : I \to [0, +\infty[$ (called the length function) which can be extended continuously by periodicity, generate from α the following curve:

$$\gamma(s) := \alpha(s) + p(s)\,\beta(s).$$

That is to say, for each $s \in I$, $\gamma(s)$ is generated by going from $\alpha(s)$ the length $p(s)$ in the direction forming an angle $\nu(s)$ with $\mathbf{t}(s)$. The curve γ will be referred to as the *generated curve*. See in Figure 3.33 an example of generated curve.

Notice that the uniqueness of $\beta(s)$ for each $\alpha(s)$ implies a non-retrograde movement of the endpoint lying in α.

Remark 3.49 (Regularity of the generated curves)**.** Suppose α to be arc-length parameterized. From the expression

$$\gamma(s) = \alpha(s) + p(s)\,\big(\cos\nu(s)\,\mathbf{t}(s) + \sin\nu(s)\,\mathbf{n}(s)\big),$$

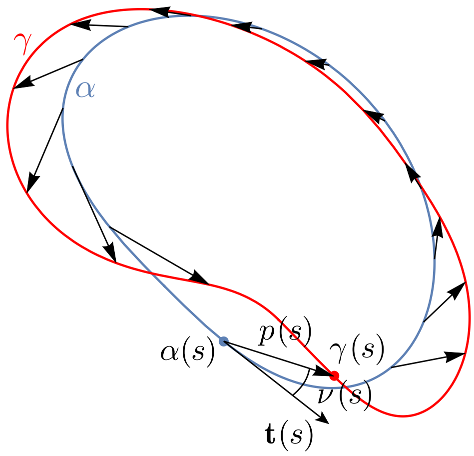

Figure 3.33: The initial curve α and its generated curve for a length function $p(s)$ and directions given by an angle function $\nu(s)$ with α.

it is easy to show that

$$\gamma'(s) = \Big(1 + p'(s)\cos\nu(s) - p(s)\sin\nu(s)\left(\nu'(s) + \kappa(s)\right)\Big)\,\mathbf{t}(s)$$
$$\Big(p'(s)\sin\nu(s) + p(s)\cos\nu(s)\left(\nu'(s) + \kappa(s)\right)\Big)\,\mathbf{n}(s).$$

Therefore,

$$\left\|\gamma'(s)\right\|^2 = \left(p'(s) + \cos\nu(s)\right)^2 + \Big(\sin\nu(s) - p(s)\left(\nu'(s) + \kappa(s)\right)\Big)^2. \tag{3.27}$$

Thus, the generated curve γ is regular if and only if

$$p'(s) + \cos\nu(s) \neq 0$$

or

$$\sin\nu(s) - p(s)\left(\nu'(s) + \kappa(s)\right) \neq 0$$

for each $s \in I$. Under some assumptions, sufficient conditions to obtain a regular generated curve can be given. This problem will be addressed in a more general setting in Chapter 5. For the particular case of parallel curves according to an angle, see Theorem 1.11. For Holditch curves, see Theorem 3.25 and Remark 5.6.

Since the functions $\beta(s)$ and $p(s)$ can be extended continuously by periodicity, the generated curve γ is closed. Also, regularity and non-double points will be assumed for γ. The general formula involving the areas of α and γ is the next.

Lemma 3.50. *Let α be a regular simple closed curve parameterized by arc-length. Suppose γ to be the generated curve from α with a length function* *p and directions given by a function β. Suppose that p and β can be continuously extended by periodicity (γ is closed). If γ is regular and simple, then*

$$\tilde{\mathcal{A}}(\gamma) = \tilde{\mathcal{A}}(\alpha) - \int_I p(s)\, \sin\nu(s)\, \mathrm{d}s + \frac{1}{2}\int_I p^2(s)\, \theta'(s)\, \mathrm{d}s, \tag{3.28}$$

where $\nu(s)$ is the oriented angle function from $\alpha'(s)$ to $\beta(s)$.

Proof. Consider $\alpha(s) = \big(x(s),\, y(s)\big)$ and adopt the notation $p := p(s)$, $\theta := \theta(s)$ $x := x(s)$, $y := y(s)$ and the same for their derivatives. The generated curve is

$$\gamma(s) = \big(x(s) + p(s)\cos\theta(s),\, y(s) + p(s)\sin\theta(s)\big).$$

Now, let's compute the area of γ:

$$\tilde{\mathcal{A}}(\gamma) = \int_I \big(x + p\cos\theta\big)\big(y' + p'\sin\theta + p\,\theta'\cos\theta\big)\, \mathrm{d}s,$$

i.e.

$$\begin{aligned}\tilde{\mathcal{A}}(\gamma) = \tilde{\mathcal{A}}(\alpha) &+ \int_I p\,(y' + x\,\theta')\cos\theta\, \mathrm{d}s + \int_I p\,p'\cos\theta\sin\theta\, \mathrm{d}s \\ &+ \int_I p^2\,\theta'\cos^2\theta\, \mathrm{d}s.\end{aligned}$$

Integrating by parts, notice that

$$\int_I p\,p'\cos\theta\sin\theta\, \mathrm{d}s = -\frac{1}{2}\int_I p^2\,\theta'\cos(2\,\theta)\, \mathrm{d}s.$$

Moreover,

$$\begin{aligned}\int_I p^2\,\theta'\cos^2\theta\, \mathrm{d}s &= \int_I p^2\,\theta'\,\frac{1+\cos(2\,\theta)}{2}\, \mathrm{d}s \\ &= \int_I \frac{p^2\,\theta'}{2}\, \mathrm{d}s + \frac{1}{2}\int_I p^2\,\theta'\cos(2\,\theta)\, \mathrm{d}s.\end{aligned}$$

Therefore,

$$\tilde{\mathcal{A}}(\gamma) = \tilde{\mathcal{A}}(\alpha) + \int_I p\,(y' + x\,\theta')\cos\theta\, \mathrm{d}s + \int_I \frac{p^2\,\theta'}{2}\, \mathrm{d}s.$$

Now,

$$\begin{aligned}\int_I p\,(y' + x\,\theta')\cos\theta\, \mathrm{d}s &= \int_I p\,\big(y'\cos\theta + x\,\theta'\cos\theta\big)\, \mathrm{d}s \\ &= \int_I p\,\big(y'\cos\theta + x\,\theta'\cos\theta - x'\sin\theta + x'\sin\theta\big)\, \mathrm{d}s \\ &= \int_I p\,\big(y'\cos\theta - x'\sin\theta\big)\, \mathrm{d}s.\end{aligned}$$

So, the formula which is obtained is

$$\tilde{\mathcal{A}}(\gamma) = \tilde{\mathcal{A}}(\alpha) + \int_I p(s)\,\lambda(s)\,\mathrm{d}s + \frac{1}{2}\int_I p^2(s)\,\theta'(s)\,\mathrm{d}s, \tag{3.29}$$

where

$$\lambda(s) = y'(s)\cos\theta(s) - x'(s)\sin\theta(s).$$

Notice that $\theta(s) = \nu(s) + \sigma(s)$, where $\sigma(s)$ is an oriented angle function from the positive OX axis to the tangent vector $\mathbf{t}(s)$ of α at $\alpha(s)$. Hence,

$$\lambda(s) = \cos\big(\nu(s) + \sigma(s)\big)\,y'(s) - \sin\big(\nu(s) + \sigma(s)\big)\,x'(s),$$

but $y'(s) = \sin\sigma(s)$ and $x'(s) = \cos\sigma(s)$, so simplifying we get

$$\lambda(s) = -\sin\nu(s).$$

Therefore, from (3.29) the expression of the statement is found. □

From Equation (3.28) of Lemma 3.50 some results can be deduced taking particular situations for the generated curve γ—only the case of regular and simple generated curves will be considered.

Definition 3.51 (Admissible length)**.** A constant length p is called *admissible* if the generated curve γ is regular and simple.

3.5.1 Parallel curves

If $p(s) = p$ and $\nu(s) = \omega$ are constant, then the generated curve γ is a parallel curve to α at a distance p according to an angle ω (see Figure 3.34).

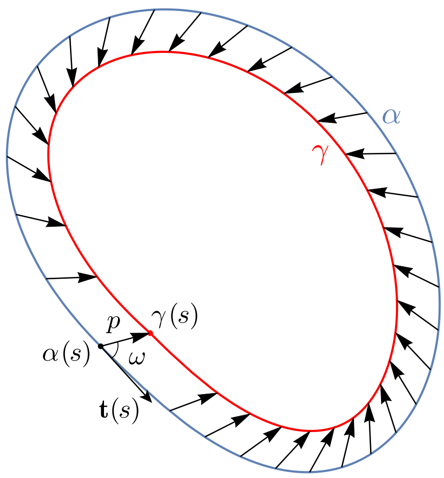

Figure 3.34: Example of a parallel curve at a constant distance p in a direction maintaining a constant angle ω with the initial curve.

Under these assumptions, the formula (3.28) from Lemma 3.50 turns into the expression below.

Theorem 3.52 (Steiner's formula for the area of a parallel curve in the plane)**.** *Let α be a positively oriented regular simple closed planar curve. Let γ be the parallel curve to α at a constant distance p maintaining a constant angle ω with α. If p is admissible, then*

$$\mathcal{A}(\gamma) = \mathcal{A}(\alpha) - p\,\mathcal{L}(\alpha)\,\sin\omega + \pi\,p^2. \tag{3.30}$$

Remark 3.53. The expression given in Theorem 3.52 holds for any angle $\omega \in \,]-\pi, \pi]$. If ω is only considered to be positive, the formal expression (3.30) would refer to an inner parallel curve. In such a case, the analogous formula for outer parallel curves could be obtained just by considering an angle $-\omega$ instead of ω in that expression, which would give:

$$\mathcal{A}(\gamma) = \mathcal{A}(\alpha) + p\,\mathcal{L}(\alpha)\,\sin\omega + \pi\,p^2.$$

The length formula for orthogonal offset curves was given in Theorem 1.8. Of course, it is also a particular case of Equation (3.27) for $p(s) = d$ and $\nu(s) = \pm\frac{\pi}{2}$, where the right hand side of such an equation reduces to a perfect square.

3.5.2 Constant width curves

If $p(s) = \ell$ is constant, $\nu(s) = \pi/2$ and it is imposed that the generated curve γ is another parameterization of α, then α is a curve of constant width ℓ (see Figure 3.35).

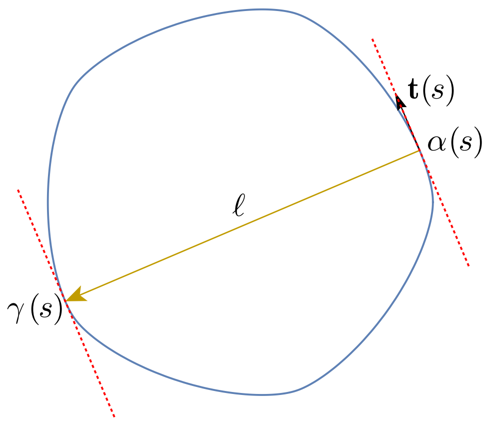

Figure 3.35: Example of a curve of constant width ℓ.

Now, Barbier's theorem (Theorem 1.19) can be proved with this approach.

Theorem 3.54 (Barbier's theorem in the plane)**.** *Let α be a regular simple closed planar curve of constant width ℓ. Then*

$$\mathcal{L}(\alpha) = \pi\,\ell.$$

Proof. Suppose α positively oriented. Since α is a curve of constant width $p(s) = \ell$, the generated curve is another parameterization for α. Thus, the formula (3.28) of Lemma 3.50 turns into

$$\mathcal{A}(\alpha) = \mathcal{A}(\alpha) - \ell\,\mathcal{L}(\alpha) + \pi\,\ell^2,$$

which can be simplified and written as in the statement. □

3.5.3 Holditch's extensions

Holditch curves are also a particular case of generated curves and Holditch's theorem can be easily obtained for the non-retrograde motion case. In fact, from Lemma 3.50 an extension of the classical statement can be given for a Holditch motion involving two initial curves instead of only one (see Figure 3.36). This extension is known as Woolhouse's theorem (see Section 2.2.2 of Chapter 2).

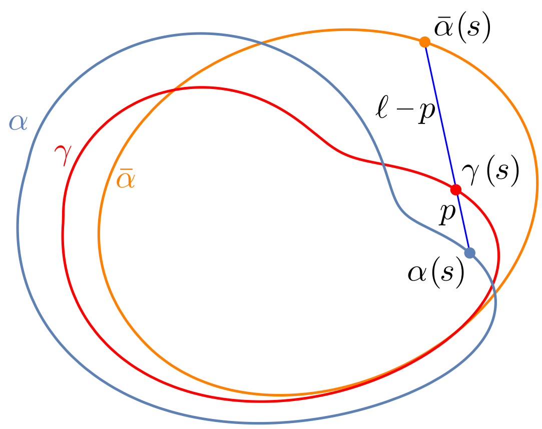

Figure 3.36: Example of a p/ℓ-Holditch curve generated by a chord of constant length ℓ moving smoothly with its endpoints on two initial curves α and $\bar{\alpha}$.

Define the *chord revolutions* as the number of positive (counterclockwise) revolutions minus the number of negative (clockwise) revolutions of the chord (seen as an indicatrix). If the chord comes back to its former position without making any full revolution, the chord revolutions are zero.

Theorem 3.55 (Holditch's theorem for two initial curves in the plane)**.** *Let α and $\bar{\alpha}$ be two closed simple and regular planar curves. Suppose that a chord of constant length $\ell > 0$ moves forwards with each of its endpoints in each curve such that n chord revolutions are done. A point dividing the chord into two parts of lengths p and $\ell - p$ will describe another closed curve γ (namely, the p/ℓ-Holditch curve). If p is admissible, then*

$$\tilde{\mathcal{A}}(\gamma) = \frac{(\ell - p)\,\tilde{\mathcal{A}}(\alpha) + p\,\tilde{\mathcal{A}}(\bar{\alpha})}{\ell} - n\,\pi\,p\,(\ell - p). \tag{3.31}$$

Proof. The setting of the statement can be given in terms of generated curves. Suppose that γ and $\bar{\alpha}$ are described from α at constant lengths p and ℓ, respectively, in the direction of the moving chord. Applying Lemma 3.50 to both generated curves, we have

$$\mathcal{A}(\gamma) = \mathcal{A}(\alpha) - p \int_I \sin\nu(s)\,\mathrm{d}s + n\,\pi\,p^2 \tag{3.32}$$

and

$$\mathcal{A}(\bar{\alpha}) = \mathcal{A}(\alpha) - \ell \int_I \sin\nu(s)\,\mathrm{d}s + n\,\pi\,\ell^2. \tag{3.33}$$

If Equation (3.32) is multiplied by ℓ and Equation (3.33) multiplied by p is subtracted, then

$$\ell\,\mathcal{A}(\gamma) - p\,\mathcal{A}(\bar{\alpha}) = (\ell - p)\,\mathcal{A}(\alpha) - n\,\pi\,p\,\ell\,(\ell - p),$$

which can be rewritten as in the statement. □

Remark 3.56. Of course, if $\bar{\alpha}$ is another parameterization for α, then the classical Holditch statement for any number n of chord revolutions is recovered:

$$\tilde{\mathcal{A}}(\alpha) - \tilde{\mathcal{A}}(\gamma) = n\,\pi\,p\,(\ell - p).$$

Remark 3.57 (Bounding the Holditch area in a variable case)**.** Let α be a strictly convex planar curve. Suppose that a chord of constant length $\ell \neq 0$ moves without retrograde motion with its endpoints on the curve α a full positive revolution. In that setting, $\alpha \circ f$ can be seen as a generated curve from α. Thus, by Lemma 3.50,

$$\int_I \sin\nu(s)\,\mathrm{d}s = \pi\,\ell.$$

Moreover, any Holditch curve can also be seen as a generated curve from α and Lemma 3.50 can be applied as well. Suppose a Holditch curve γ defined by a variable length $p(s)$ such that $0 \leq p(s) \leq \ell$ for all $s \in I$. Since $\theta'(s) > 0$ by Proposition 3.22, we have

$$\int_I p^2(s)\,\theta'(s)\,\mathrm{d}s \geq 0.$$

By Lemma 3.50 applied to the generated curve γ,

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = \int_I p(s)\sin\nu(s)\,\mathrm{d}s - \int_I p^2(s)\,\theta'(s)\,\mathrm{d}s.$$

Therefore,

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) \leq \int_I p(s)\sin\nu(s)\,\mathrm{d}s \leq \ell \int_I \sin\nu(s)\,\mathrm{d}s = \pi\,\ell^2.$$

In a few words, the area defined by a ball of radius the length of the Holditch chord is an upper bound for the Holditch area in this variable point case.

If $p(s) = p$ is constant, then the value of p which maximizes the Holditch area $\pi\, p\,(\ell - p)$ is $\ell/2$. Therefore, for any p-Holditch curve, an upper bound for the Holditch area is

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) \leq \frac{\pi\,\ell^2}{4}.$$

3.5.4 Bicycle curves

If α is the rear wheel of a bicycle (as defined in Section 1.4), then the generated curve at a constant length ℓ for tangent directions, γ, is the front wheel (see Figure 3.37). The following result is well known in the literature (see [1] or [31]) and its proof can be given as a consequence of Lemma 3.50.

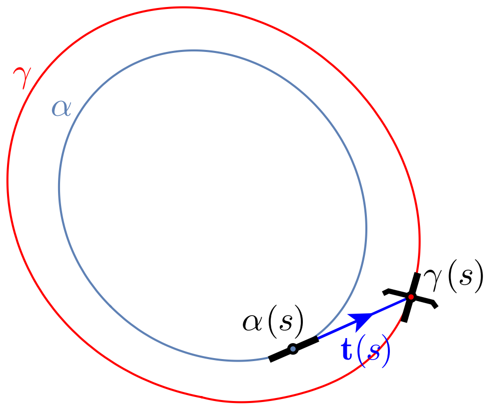

Figure 3.37: Motion of a bicycle where the rear wheel α is the initial curve and the front wheel γ the generated one.

Theorem 3.58. *If α_R and α_F are two regular simple closed planar curves describing the rear and the front wheel, respectively, of a bicycle of constant length ℓ, then*

$$\tilde{\mathcal{A}}(\alpha_F) - \tilde{\mathcal{A}}(\alpha_R) = \pi\,\ell^2.$$

Proof. Consider α_F to be the generated curve from α_R at a distance ℓ in the tangent directions (i.e. $\nu(s) = 0$). By Lemma 3.50,

$$\tilde{\mathcal{A}}(\alpha_F) = \tilde{\mathcal{A}}(\alpha_R) + \frac{1}{2}\,\ell^2 \int_I \theta'(s)\,\mathrm{d}s.$$

Since α_R and α_F are simple, we have that

$$\int_I \theta'(s)\,\mathrm{d}s = 2\,\pi$$

and the result is deduced. $\square$

Chapter 4

An introduction to non-Euclidean geometry

First, a brief introduction to quadratic forms is given. A quadratic space is defined so that the models of elliptic and hyperbolic geometry raise naturally from it as Riemannian manifolds. The framework is studied in detail and some properties are derived. The given construction allows us to work at the same time with both models for non-Euclidean geometry. In the last section, the non-Euclidean versions of parallel curves, constant width curves and bicycle tire-tracks are defined.

4.1 Preliminaries on quadratic forms

In this section, a brief introduction to quadratic forms will be given with the aim to define later a quadratic space. As main references for his section, see [8], [40], [50], [67] or [86]. At first, any field K is allowed, but only quadratic forms over the field $\mathbb{R}$ will be used later.

4.1.1 Definition as a polynomial

Definition 4.1 (Quadratic form)**.** A *n-ary quadratic form over a field K* is a homogeneous polynomial of degree 2 in n variables with coefficients in K, i.e. a polynomial

$$\mathcal{Q}(x_1, \dots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \tag{4.1}$$

with $a_{ij} \in K$.

If $\mathbf{x} = (x_1, \dots, x_n)^T \in K^n$ and $A = (a_{ij}) \in \mathcal{M}_n(K)$, with $\mathcal{M}_n(K)$ being the set of $n \times n$ matrices over K, (4.1) can be rewritten as

$$\mathcal{Q}(\mathbf{x}) = \mathbf{x}^T A \, \mathbf{x}.$$

The matrix A is called the *associated matrix of the quadratic form $\mathcal{Q}$.*

Recall now the definition of the characteristic of a ring.

Definition 4.2 (Characteristic of a ring)**.** Let $(R, +, \cdot)$ be a ring. The *characteristic of R*, $\operatorname{char}(R)$, is defined to be the smallest number of times the multiplicative element 1 must be used in a sum to get the additive identity 0. A ring is said to have *characteristic zero* if this sum never reaches the additive identity 0.

If we adopt the convention $a_{ij} = a_{ji}$, note that for fields K of characteristic 2, all the cross product terms of (4.1) disappear. The usual procedure is to require $a_{ij} = a_{ji}$ if $\operatorname{char}(K) \neq 2$ and restrict to a_{ij}, $i \leq j$, if $\operatorname{char}(K) = 2$.

Let's focus now on fields of characteristic not equal to 2. Thus, the following definition makes sense.

Definition 4.3 (Polarization of a quadratic form)**.** Let K be a field of characteristic not equal to 2. The *polarization of a quadratic form $\mathcal{Q}$ over K* is the function $\mathcal{B}_{\mathcal{Q}} : K^n \times K^n \to K$ defined by

$$\mathcal{B}_{\mathcal{Q}}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \big(\mathcal{Q}(\mathbf{x} + \mathbf{y}) - \mathcal{Q}(\mathbf{x}) - \mathcal{Q}(\mathbf{y}) \big),$$

for $\mathbf{x}, \mathbf{y} \in K^n$.

Note that

$$\mathcal{B}_{\mathcal{Q}}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i y_j, \tag{4.2}$$

which can be written as

$$\mathcal{B}_{\mathcal{Q}}(\mathbf{x},\mathbf{y}) = \mathbf{x}^T A\,\mathbf{y} = \mathbf{y}^T A\,\mathbf{x}.$$

Moreover, notice that

$$\mathcal{B}_{\mathcal{Q}}(\mathbf{x},\mathbf{x}) = \frac{1}{2}\big(\mathcal{Q}(2\,\mathbf{x}) - 2\,\mathcal{Q}(\mathbf{x})\big) = \mathcal{Q}(\mathbf{x}).$$

Immediately, from Equation (4.2), we have the following.

Proposition 4.4. *The polarization of a quadratic form over a field of characteristic not equal to 2 is a symmetric bilinear form.*

From Proposition 4.4, we may refer to $\mathcal{B}_{\mathcal{Q}}$ as the *associated bilinear form of* $\mathcal{Q}$.

Conversely, given a field K such that $\mathrm{char}(K) \neq 2$, any symmetric bilinear form $\mathcal{B} : K^n \times K^n \to K$ defines a function $\mathcal{Q} : K^n \to K$ by

$$\mathcal{Q}(\mathbf{x}) = \mathcal{B}(\mathbf{x},\mathbf{x}),$$

which can be viewed as a polynomial over K and, thus, as a quadratic form. We will say that $\mathcal{Q}$ is the *associated quadratic form of* $\mathcal{B}$.

Readily, both processes described before are inverses one of another, i.e., given a quadratic form $\mathcal{Q}$ over K, the associated quadratic form of the polarization of $\mathcal{Q}$ is $\mathcal{Q}$. In a few words, for characteristic 2, the theory of symmetric bilinear forms and of quadratic forms in n variables are essentially identical.

Remark 4.5. Let K be a field of characteristic 2. In this case, we have that $2 = 1 + 1 = 0$ is not a unit and Definition 4.3 does not make sense. In this case, given a quadratic form $\mathcal{Q}$ over K, we can define a symmetric bilinear form

$$\mathcal{B}'(\mathbf{x},\mathbf{y}) = \mathcal{Q}(\mathbf{x}+\mathbf{y}) - \mathcal{Q}(\mathbf{x}) - \mathcal{Q}(\mathbf{y}),$$

for $\mathbf{x},\mathbf{y} \in K^n$. Nevertheless, from $\mathcal{B}'$ we cannot recover $\mathcal{Q}$ since $\mathcal{B}'(\mathbf{x},\mathbf{x}) = 0$ for all $\mathbf{x} \in K^n$. In this case, the theory of quadratic forms is different to the theory of symmetric bilinear forms and it deserves a separated study.

Definition 4.6 (Equivalent quadratic forms)**.** Two n-ary quadratic forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ over a field K are *equivalent* if there exists a non-singular linear transformation $C \in \mathrm{GL}_n(K)$, where $\mathrm{GL}_n(K)$ is the general linear group, such that

$$\mathcal{Q}_2(\mathbf{x}) = \mathcal{Q}_1(C\,\mathbf{x})$$

for all $\mathbf{x} \in K^n$.

Since every symmetric matrix is orthogonally diagonalizable, it follows that any quadratic form over a field K with $\mathrm{char}(K) \neq 2$ is equivalent to a *diagonal form.* The reader can find the proof of the following theorem in [8].

Theorem 4.7. *Every quadratic form $\mathcal{Q}$ in n variables over a field K of characteristic not equal to 2 is equivalent to a diagonal form*

$$\mathcal{Q}(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2,$$

where $a_i \in K$.

If $\mathcal{Q}$ is a quadratic form in n variables, to avoid equivalent quadratic forms in less than n variables, non-degenerated quadratic forms can be considered.

Definition 4.8 (Non-degenerate quadratic form)**.** A quadratic form is said to be *non-degenerated* if its associated matrix has determinant not equal to zero.

We will mainly work with quadratic forms over the field $\mathbb{R}$. In such a case, Theorem 4.7 can be specified a little.

Corollary 4.9. *Every non-degenerated quadratic form in n variables over $\mathbb{R}$ is equivalent to a diagonal form*

$$\mathcal{Q}(x_1, \ldots, x_n) = a_1\, x_1^2 + \cdots + a_n\, x_n^2,$$

where $a_i \in \{-1, 1\}$.

Proof. By Theorem 4.7, we have that any quadratic form $\mathcal{Q}$ over $\mathbb{R}$ is equivalent to a form

$$\mathcal{Q}_2(x_1, \ldots, x_n) = a_1\, x_1^2 + \cdots + a_n\, x_n^2,$$

with $a_i \in \mathbb{R}$. Since $\mathcal{Q}$ (and thus $\mathcal{Q}_2$) is non-degenerated, we have $a_i \neq 0$ (otherwise, the determinant of the diagonal matrix would be zero). Now, make the (invertible) variable change $x_i \mapsto \frac{1}{\sqrt{|a_i|}}\, x_i$. With that, we have that $\mathcal{Q}_2$ is equivalent to the form

$$\mathcal{Q}_3(x_1, \ldots, x_n) = \mathrm{sgn}(a_1)\, x_1^2 + \cdots + \mathrm{sgn}(a_n)\, x_n^2,$$

which proves the theorem. Some authors apply a permutation to write all the positive terms together apart from the negative ones. $\square$

Remark 4.10. Corollary 4.9 is usually stated for any quadratic form (not only the non-degenerated ones). In that case, $a_i \in \{-1, 0, 1\}$. This result is known as *Sylvester's law of inertia.*

Definition 4.11 (Signature of a quadratic form)**.** Let $\mathcal{Q}$ be a quadratic form in n variables over $\mathbb{R}$. The *signature of $\mathcal{Q}$* is defined as the difference between the number of positive and the number of negative coefficients which we obtain when $\mathcal{Q}$ is reduced to an equivalent diagonal form.

Denote by s the signature of a real non-degenerated quadratic form in n variables. If P and N are the number of positive and negative, respectively, coefficients of the diagonal form, then

$$P + N = n \quad \text{and} \quad P - N = s.$$

Therefore,

$$P = \frac{n+s}{2} \quad \text{and} \quad N = \frac{n-s}{2}.$$

Theorem 4.12. *Two real non-degenerated quadratic forms are equivalent if and only if they have the same signature.*

The proof of this theorem and a further detailed discussion can be found in [8].

4.1.2 Coordinate-free definition

It is standard to define a quadratic form in a coordinate-free way. It is more advantageous because it allows one to use more easily the theory of vector spaces and linear maps.

Definition 4.13 (Quadratic form: alternative definition)**.** Let V be a finite dimensional vector space over a field K. A *quadratic form on V* is a map $\mathcal{Q} : V \to K$ such that

1. $\mathcal{Q}(c\,\mathbf{v}) = c^2\mathcal{Q}(\mathbf{v})$ for all $\mathbf{v} \in V$ and $c \in K$, and
2. the symmetric map $\mathcal{B}_\mathcal{Q} : V \times V \to K$ defined by
$$\mathcal{B}_\mathcal{Q}(\mathbf{u}, \mathbf{v}) = \mathcal{Q}(\mathbf{u} + \mathbf{v}) - \mathcal{Q}(\mathbf{u}) - \mathcal{Q}(\mathbf{v})$$
is bilinear.

The symmetric bilinear form $\mathcal{B}_\mathcal{Q}$ is called the *polar form of $\mathcal{Q}$*.

Note that here, $\mathcal{B}_\mathcal{Q}(\mathbf{u}, \mathbf{u}) = \mathcal{Q}(2\,\mathbf{u}) - 2\,\mathcal{Q}(\mathbf{u}) = 2\,\mathcal{Q}(\mathbf{u})$ for any $\mathbf{u} \in V$.

Definition 4.14 (Quadratic space)**.** A pair $(V, \mathcal{Q})$ consisting of a vector space V over a field K and a quadratic form $\mathcal{Q}$ on V is called a *quadratic space.*

Proposition 4.15. *Definitions 4.1 and 4.13 of a quadratic form are equivalent.*

Proof. Any quadratic form on the sense of Definition 4.1 readily satisfies the properties of Definition 4.13, with $V = K^n$ being the vector space. Conversely, let $\mathcal{Q}$ be a quadratic form on a n-dimensional vector space V and let

$\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis of V. Suppose that $\mathbf{x} = \sum_{i=1}^n x_i\, \mathbf{u}_i \in V$ and name $c_{ij} = \mathcal{B}(\mathbf{u}_i, \mathbf{u}_j)$. If $n = 1$,

$$\mathcal{Q}(\mathbf{x}) = \mathcal{Q}(x_1\, \mathbf{u}_1) = x_1^2\, \mathcal{Q}(\mathbf{u}_1) = \frac{c_{11}}{2}\, x_1^2,$$

which is a quadratic form in the sense of Definition 4.1. If $n = 2$, by definition of the polar form $\mathcal{B}_{\mathcal{Q}}$:

$$\begin{aligned}\mathcal{Q}(\mathbf{x}) &= \mathcal{Q}(x_1\, \mathbf{u}_1 + x_2\, \mathbf{u}_2) = \mathcal{B}(x_1\, \mathbf{u}_1, x_2\, \mathbf{u}_2) + \mathcal{Q}(x_1\, \mathbf{u}_1) + \mathcal{Q}(x_2\, \mathbf{u}_2)\\ &= c_{12}\, x_1\, x_2 + \frac{c_{11}}{2}\, x_1^2 + \frac{c_{22}}{2}\, x_2^2,\end{aligned}$$

which, again, is a quadratic form satisfying Definition 4.1. Proceed by induction on n. Similarly, we have

$$\begin{aligned}\mathcal{Q}(\mathbf{x}) &= \mathcal{Q}\left(\sum_{i=1}^{n-1} x_i\, \mathbf{u}_i + x_n\, \mathbf{u}_n\right)\\ &= \mathcal{Q}\left(\sum_{i=1}^{n-1} x_i\, \mathbf{u}_i\right) + \mathcal{Q}(x_n\, \mathbf{u}_n) + \mathcal{B}_{\mathcal{Q}}\left(\sum_{i=1}^{n-1} x_i\, \mathbf{u}_i,\, x_n\, \mathbf{u}_n\right).\end{aligned}$$

By bilinearity,

$$\mathcal{B}_{\mathcal{Q}}\left(\sum_{i=1}^{n-1} x_i\, \mathbf{u}_i,\, x_n\, \mathbf{u}_n\right) = \sum_{i=1}^{n-1} c_{in}\, x_i\, x_n.$$

Also, $\mathcal{Q}(x_n\, \mathbf{u}_n) = \frac{c_{nn}}{2}\, x_n^2$ and, by induction hypothesis,

$$\mathcal{Q}\left(\sum_{i=1}^{n-1} x_i\, \mathbf{u}_i\right)$$

is a homogeneous polynomial of degree 2. Therefore, $\mathcal{Q}(\mathbf{x})$ can be seen as a quadratic polynomial in the linear coordinates x_i (with $c_{ij} \in K$ uniquely determined by $\mathcal{Q}$ and the choice of the basis). $\square$

4.2 Introduction to non-Euclidean geometry

Here we give a brief introduction to non-Euclidean geometry. The interested reader can see [85], [83], [15], [75], [16], [78] and [59]. It is also worth to take a look at the nice introduction in [76] to the hyperboloid model of hyperbolic geometry.

First, the setting will be introduced as a real quadratic space and a 2-dimensional Riemannian manifold of constant curvature will raise naturally from it. Among the four possible non-degenerated quadratic forms which are not equivalent (see Theorem 4.12), only two will be considered in the following section. The reason will be given later in Remark 4.22.

4.2.1 Quadratic form of the space and some definitions

Let K be a non-zero real constant. Consider the quadratic space $\left(\mathbb{R}^3, \mathcal{Q}_K\right)$ defined by the quadratic form

$$\mathcal{Q}_K(x_1, x_2, x_3) = \operatorname{sgn}(K)\, x_1^2 + x_2^2 + x_3^2.$$

For now, only the sign of K will be relevant, not its absolute value. Nevertheless, the quantity K is important since it will be interpreted later as a curvature.

The polar form of $\mathcal{Q}_K$ is the bilinear map

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\left(\mathcal{Q}_K(\mathbf{u}+\mathbf{v}) - \mathcal{Q}_K(\mathbf{u}) - \mathcal{Q}_K(\mathbf{v})\right)$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Explicitly, as seen in Equation (4.2), if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) = \operatorname{sgn}(K)\, u_1 v_1 + u_2 v_2 + u_3 v_3.$$

In fact, if $K > 0$, it corresponds to the usual *scalar product of* $\mathbb{R}^3$ and if $K < 0$ to the so-called *Lorentzian inner product*[1] *of* $\mathbb{R}^{1,2}$. The *norm associated with* $\mathcal{B}$ is the function $\|\cdot\| : \mathbb{R}^3 \to \mathbb{C}$ defined by

$$\|\mathbf{u}\| = \sqrt{\mathcal{B}(\mathbf{u}, \mathbf{u})} = \sqrt{\operatorname{sgn}(K)\, u_1^2 + u_2^2 + u_3^2}.$$

The value of this norm is either positive, zero or positive purely imaginary. Vectors of $\mathbb{R}^3$ can be classified depending on such a value:

- If $\|\mathbf{v}\| = 0$, then $\mathbf{v}$ is *light-like.*
- If $\|\mathbf{v}\| > 0$, then $\mathbf{v}$ is *space-like.*

[1] Although it is standard to use the term “inner product”, it is actually not an inner product in the usual sense because it is not positive-definite.

- If $\|\mathbf{v}\|$ is imaginary, then $\mathbf{v}$ is *time-like*. In such a case, denote its absolute value (modulus) by $|||\cdot|||$.

A *vector subspace* V *of* $\mathbb{R}^3$ is said to be

- *space-like* if every non-zero vector in V is space-like,
- *time-like* if V has a time-like vector, or
- *light-like*, otherwise.

Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we define *generalized cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ as

$$\mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v} := \begin{pmatrix} \operatorname{sgn}(K) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\mathbf{u} \wedge \mathbf{v}),$$

where $\wedge$ denotes the usual cross product. In coordinates, if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v} := \left(\operatorname{sgn}(K) \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

Note, in fact, that when $K > 0$, $\hat{\wedge}$ is the usual *cross product of* $\mathbb{R}^3$ and when $K < 0$, $\hat{\wedge}$ is the *Lorentzian cross product of* $\mathbb{R}^{1,2}$. The cross product $\hat{\wedge}$ satisfies similar properties as the usual one as is stated below (see [75] for more details).

Proposition 4.16. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and let* $\alpha : I \to \mathbb{R}^3$ *and* $\beta : I \to \mathbb{R}^3$ *be two functions defined on a real open interval* I. *Then*

1. $\mathcal{B}(\mathbf{u}, \mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v}) = \mathcal{B}(\mathbf{v}, \mathbf{u} \mathbin{\hat{\wedge}} \mathbf{v}) = 0.$
2. $\big(\alpha(t) \mathbin{\hat{\wedge}} \beta(t)\big)' = \alpha'(t) \mathbin{\hat{\wedge}} \beta(t) + \alpha(t) \mathbin{\hat{\wedge}} \beta'(t).$
3. $\mathcal{B}(\mathbf{u},\, \mathbf{v} \mathbin{\hat{\wedge}} \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}).$
4. $\mathbf{u} \mathbin{\hat{\wedge}} (\mathbf{v} \mathbin{\hat{\wedge}} \mathbf{w}) = \operatorname{sgn}(K)\big(\mathbf{v}\, \mathcal{B}(\mathbf{u}, \mathbf{w}) - \mathbf{w}\, \mathcal{B}(\mathbf{u}, \mathbf{v})\big).$

Henceforth we will denote simply by $\wedge$ the generalized cross product $\hat{\wedge}$. This abuse of notation is justified by the fact that both operators have similar properties, taking care only on small differences involving the sign of K.

An *orthogonal basis* is a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ such that $\mathcal{B}(\mathbf{u}_i, \mathbf{u}_j) = 0$ for $i \neq j$. If $K > 0$, it is the common definition of an *orthogonal basis of* $\mathbb{R}^3$ and when $K < 0$ it corresponds to the definition of a *Lorentz orthogonal basis of* $\mathbb{R}^{1,2}$. From the definition, it follows that each Lorentz orthogonal basis of $\mathbb{R}^3$ consists of one time-like vector and two space-like vectors (see [3]).

An *orthonormal basis* will be understood as a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ such that $\mathcal{B}(\mathbf{u}_1, \mathbf{u}_1) = \operatorname{sgn}(K)$ and $\mathcal{B}(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$ otherwise. If $K > 0$ it is the common definition of an *orthonormal basis of* $\mathbb{R}^3$ and when $K < 0$ it corresponds to the definition of a *Lorentz orthonormal basis of* $\mathbb{R}^{1,2}$.

Proposition 4.17. *If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal vectors such that $\|\mathbf{u}_1\|^2 = \operatorname{sgn}(K)$ and $\|\mathbf{u}_2\|^2 = 1$, then*

$$\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\},$$

where $\mathbf{u}_3 = \mathbf{u}_1 \wedge \mathbf{u}_2$, is an orthonormal basis and their vectors satisfy:

$$\begin{aligned}
\mathbf{u}_1 \wedge \mathbf{u}_2 &= \mathbf{u}_3, \\
\mathbf{u}_2 \wedge \mathbf{u}_3 &= \operatorname{sgn}(K)\, \mathbf{u}_1, \\
\mathbf{u}_3 \wedge \mathbf{u}_1 &= \mathbf{u}_2.
\end{aligned}$$

Proof. From the definition of the cross product, $\mathbf{u}_3$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. Their cross product is a unit vector (use Proposition 4.16):

$$\begin{aligned}
\|\mathbf{u}_1 \wedge \mathbf{u}_2\|^2 &= \mathcal{B}(\mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{u}_1 \wedge \mathbf{u}_2) = \det(\mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{u}_1, \mathbf{u}_2) \\
&= \det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_1 \wedge \mathbf{u}_2) = \mathcal{B}\big(\mathbf{u}_1, \mathbf{u}_2 \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2)\big) \\
&= \operatorname{sgn}(K)\, \mathcal{B}(\mathbf{u}_1, \mathbf{u}_1)\, \mathcal{B}(\mathbf{u}_2, \mathbf{u}_2) = \operatorname{sgn}(K)^2 = 1.
\end{aligned}$$

The relations between the three vectors are obtained with easy calculations. □

4.2.2 The Riemannian manifold M^K

In this section, a 2-dimensional Riemannian manifold of constant curvature will be defined and studied. The idea is to use it as a model of non-Euclidean geometry. For a positive curvature, the sphere is recovered as the model for spherical geometry and for a negative curvature, the hyperboloid model for hyperbolic geometry is considered.

Definition of the manifold

Given $K \neq 0$, a differentiable manifold can be defined as follows:

$$M^K = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \mathcal{Q}_K(x_1, x_2, x_3) = \frac{1}{K},\ x_1 \in D_K \right\},$$

where

$$\mathcal{Q}_K(x_1, x_2, x_3) = \operatorname{sgn}(K)\, x_1^2 + x_2^2 + x_3^2$$

and

$$D_K = \begin{cases}]0, +\infty[& \text{if } K < 0, \\ \mathbb{R} & \text{if } K > 0. \end{cases}$$

Note that if $K<0$, $x_1 \geq 1/\sqrt{|K|}$. An atlas of M^K can be defined using the parametric surface $\mathbf{x}^K : U \times V \to \mathbb{R}^3$ given by

$$\mathbf{x}^K(u,v) = \left(\frac{\cos\big(\sqrt{K}\,v\big)}{\sqrt{|K|}}, \frac{\sin(u)\sin\big(\sqrt{K}\,v\big)}{\sqrt{K}}, \frac{\cos(u)\sin\big(\sqrt{K}\,v\big)}{\sqrt{K}} \right), \tag{4.3}$$

where

$$U = \left]0, 2\pi\right[\quad \text{and} \quad V = \begin{cases} \left]0, \pi/\sqrt{K}\right[, & \text{if } K>0, \\ \left]0, +\infty\right[& \text{if } K<0. \end{cases}$$

Remark 4.18. Define $R^2 = \frac{1}{|K|}$.

If $K>0$, the manifold M^K is the 2-*dimensional sphere of radius* R:

$$\mathbb{S}^2(K) := \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = R^2\big\}.$$

It can be parameterized by

$$\mathbf{x}_S(u,v) = R\,\big(\cos(v/R),\, \sin(u)\sin(v/R),\, \cos(u)\sin(v/R)\big),$$

for $u \in \left]0,2\pi\right[$ and $v \in \left]0, \pi R\right[$.

If $K<0$, M^K is the positive sheet of the 2-*dimensional two-sheeted hyperboloid with vertex at a distance R from the origin*:

$$\mathbb{H}^2(K) := \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : -x_1^2 + x_2^2 + x_3^2 = -R^2,\ x_1 > 0\big\}.$$

The corresponding parameterization is

$$\mathbf{x}_H(u,v) = R\,\big(\cosh(v/R),\, \sin(u)\sinh(v/R),\, \cos(u)\sinh(v/R)\big),$$

for $u \in \left]0,2\pi\right[$ and $v>0$.

Tangent space

Since M^K is a subset of $\mathbb{R}^3$, the tangent space to M^K at a point $p \in M^K$, T_pM^K, is a 2-dimensional vector subspace of $\mathbb{R}^3$.

Proposition 4.19. *The tangent space of M^K at a point $p \in M^K$ is*

$$T_pM^K = \big\{\mathbf{v} \in \mathbb{R}^3 : \mathcal{B}(p,\mathbf{v}) = 0\big\}.$$

Proof. If $\mathbf{v} \in T_pM^K$, then there exists a differentiable curve $\alpha : I \to M^K$, $0 \in I$, such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. We have that $\mathcal{B}\big(\alpha(t), \alpha(t)\big) = \frac{1}{K}$ for all $t \in I$. Differentiating this, we get $\mathcal{B}\big(\alpha'(t), \alpha(t)\big) = 0$. Therefore, $\mathcal{B}(p,\mathbf{v}) = \mathcal{B}(\alpha(0), \alpha'(0)\big) = 0$, which proves the first inclusion.

Conversely, given $\mathbf{v} \in \mathbb{R}^3$ such that $\mathcal{B}(p,\mathbf{v}) = 0$, define a curve $\alpha : I \to M^K$ as $\alpha = \mathbf{x}^K \circ \beta$, where $\beta(t) = q + t\,\mathbf{v}$ and $q = \big(\mathbf{x}^K\big)^{-1}(p)$. This curve verifies $\alpha(0) = p$ and

$$\alpha'(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\mathbf{x}^K \circ \beta\big) = D\mathbf{x}^K_q(\mathbf{v}) = D\mathbf{x}^K_{\beta(0)}\big(\beta'(0)\big) \in T_{\alpha(0)}M^K.$$

This second inclusion could also be proved taking into account the other inclusion and the fact that the dimension of T_pM^K is 2. $\square$

The Riemannian metric

Given $p \in M^K$, define $g_p : T_pM^K \times T_pM^K \to \mathbb{R}$ in terms of the bilinear function:

$$g_p(\mathbf{v}, \mathbf{w}) := g(\mathbf{v}, \mathbf{w})(p) = \mathcal{B}(\mathbf{v}, \mathbf{w}).$$

That is to say, g_p is the restriction of $\mathcal{B}$ to tangent vectors of T_pM^K.

Proposition 4.20. $\left(M^K, g\right)$ *is a Riemannian manifold.*

Proof. We must show that g_p is a Riemannian metric at p for M^K. It is trivially a bilinear symmetric map. Let's show that it is positive-definite, i.e. that $g_p(\mathbf{v}, \mathbf{v}) \geq 0$ for all $\mathbf{v} \in T_pM^K$. Suppose $K < 0$ (the case $K > 0$ is obvious). Let $p = (p_1, p_2, p_3) \in M^K$ and $\mathbf{v} = (v_1, v_2, v_3) \in T_pM^K$. Denote $\tilde{p} = (p_2, p_3)$ and $\tilde{\mathbf{v}} = (v_2, v_3)$. From the condition $\mathcal{B}(p, \mathbf{v}) = 0$ we deduce

$$v_1 = \frac{\langle \tilde{p}, \tilde{\mathbf{v}} \rangle}{p_1}.$$

Note that $p_1 > 0$ because $K < 0$. Thus,

$$\mathcal{B}(\mathbf{v}, \mathbf{v}) = -\frac{\langle \tilde{p}, \tilde{\mathbf{v}} \rangle^2}{p_1^2} + \|\tilde{\mathbf{v}}\|^2 = \frac{\|\tilde{\mathbf{v}}\|^2}{p_1^2} \left(p_1^2 - \|\tilde{p}\|^2 \cos^2 \delta\right),$$

where δ is the angle between the vectors $\tilde{p}$ and $\tilde{\mathbf{v}}$. Now, since

$$\|\tilde{p}\|^2 = \frac{1}{K} + p_1^2,$$

we have that

$$\mathcal{B}(\mathbf{v}, \mathbf{v}) = \frac{\|\tilde{\mathbf{v}}\|^2}{p_1^2} \left(p_1^2 \sin^2 \delta - \frac{1}{K} \cos^2 \delta\right) \geq 0,$$

which completes the proof. □

Once the structure of a Riemannian manifold has been constructed, the Gauss curvature (as its sectional curvature) can be computed. The coefficients of the first fundamental form are defined as usual but with the metric g. In particular, note that the classical formula given in [20], p. 237, can be used:

Proposition 4.21. *The Gauss curvature of M^K is K.*

Proof. Since the parameterization $\mathbf{x}^K$ is orthogonal ($g_{12} = 0$), the Gauss curvature can be computed by

$$-\frac{1}{2\sqrt{g_{11}\, g_{22}}} \left(\left(\frac{g_{11,2}}{\sqrt{g_{11}\, g_{22}}} \right)_2 + \left(\frac{g_{22,1}}{\sqrt{g_{11}\, g_{22}}} \right)_1 \right).$$

A straightforward computation shows that it is exactly K. □

Remark 4.22. As it has been said at the beginning of the section, only two of the four non-degenerated quadratic forms which are not equivalent have been considered. Each one together with the metric induced by the associated bilinear form yields a different kind of constant curvature geometry: the spherical and the hyperbolic.

What about the other possibilities? Recall that our manifold is defined by an equality of the kind:

$$\mathcal{Q}(x_1, x_2, x_3) = \frac{1}{K}, \tag{4.4}$$

where K is a constant. The four possible different quadratic forms are:

$$\begin{aligned}
\mathcal{Q}_1(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2,\\
\mathcal{Q}_2(x_1, x_2, x_3) &= -x_1^2 + x_2^2 + x_3^2,\\
\mathcal{Q}_3(x_1, x_2, x_3) &= -x_1^2 - x_2^2 + x_3^2,\\
\mathcal{Q}_4(x_1, x_2, x_3) &= -x_1^2 - x_2^2 - x_3^2.
\end{aligned}$$

On the one hand, the quadratic form $\mathcal{Q}_1$ can satisfy an equation as (4.4) only for the case of $K > 0$. On the other hand, the same for $\mathcal{Q}_4$ is only possible for $K < 0$, which yields the same model (of spherical geometry) by multiplying the entire equation by a minus sign. Similarly, the four possibilities given by the quadratic forms $\mathcal{Q}_3$ and $\mathcal{Q}_4$ together with a positive or negative K are reduced to only two:

$$-x_1^2 + x_2^2 + x_3^2 = \frac{1}{K}, \quad \text{for } K < 0,$$

and

$$-x_1^2 + x_2^2 + x_3^2 = \frac{1}{K}, \quad \text{for } K > 0.$$

The first case has been considered with the Lorentz metric and it produces the hyperboloid model (as a Riemannian manifold) of hyperbolic geometry (with one of the sheets of the two-sheeted hyperboloid). The second case represents a one-sheeted hyperboloid. The reason that it has not been considered is because in such a case the Lorentz metric turns out to be non-definite positive and thus it cannot be seen as a Riemannian manifold. A similar discussion on this is considered in [5].

Lengths of curves

The *length* of a curve $\alpha : I \to M$ in a Riemannian manifold (M, g) is defined by

$$\mathcal{L}(\alpha) = \int_I \sqrt{\left| g_{\alpha(t)}\big(\alpha'(t), \alpha'(t)\big) \right|}\ \mathrm{d}t.$$

Given a regular curve $\alpha : I \to M^K$, we have that α' is always a space-like vector. Hence, $\|\alpha'(t)\| > 0$ for all $t \in I$ and the *length of* α *in* M^K can be written as

$$\mathcal{L}(\alpha) = \int_I \|\alpha'(t)\|\ \mathrm{d}t.$$

Definition 4.23 (Arc-length parameterization in M^K)**.** A curve $\alpha : I \to M^K$ is said to be parameterized by *arc length* if $\|\alpha'(t)\| = 1$ for all $t \in I$.

Normal vector

Since M^K is embedded in $\mathbb{R}^3$, the notion of normal vector to M^K at a point of the manifold makes sense.

Definition 4.24 (Normal vector to M^K)**.** Let $\mathbf{x}$ be a system of coordinates of M^K at a point $p \in M^K$ and suppose $\mathbf{x}(u, v) = p$. The *normal vector to M^K at the point* p is defined by

$$\mathbf{N}^{\mathbf{x}}(u, v) := \operatorname{sgn}(K)\, \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|||\mathbf{x}_u \wedge \mathbf{x}_v|||}(u, v).$$

The sign of K chosen in this definition is just a convention to shorten the notation later.

Proposition 4.25. *Let* $\mathbf{x}$ *be a system of coordinates of* M^K *at* $p = \mathbf{x}(u_0, v_0) \in M^K$*. The normal vector* $\mathbf{N}^{\mathbf{x}}$ *to* M^K *at* p *verifies:*

$$\mathbf{N}^{\mathbf{x}}(u_0, v_0) = \pm \frac{p}{|||p|||} = \pm\sqrt{|K|}\, p.$$

Proof. Suppose $\mathbf{x}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$. Since it is a chart of M^K, it must satisfy

$$\operatorname{sgn}(K)\, x^2(u, v) + y^2(u, v) + z^2(u, v) = \frac{1}{K}.$$

Differentiating this expression with respect to u and with respect to v, two equations are found:

$$\mathcal{B}\big(\mathbf{x}(u, v), \mathbf{x}_u(u, v)\big) = \mathcal{B}\big(\mathbf{x}(u, v), \mathbf{x}_v(u, v)\big) = 0.$$

This means that $\mathbf{x}(u, v)$ is orthogonal to the vectors $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$. Hence, $\mathbf{x}(u, v)$ must be parallel to $(\mathbf{x}_u \wedge \mathbf{x}_v)(u, v)$. In particular, p must be parallel to $(\mathbf{x}_u \wedge \mathbf{x}_v)(u_0, v_0)$. Therefore, their normalizations are equal up to a sign (this proves the first equality). For the second equality, just notice that since $\|p\| = 1/\sqrt{K}$, we have $|||p||| = 1/\sqrt{|K|}$. $\square$

Remark 4.26. The normal vector with the chart $\mathbf{x}^K$ defined in (4.3) equals the position vector:

$$\mathbf{N}^{\mathbf{x}^K}(u_0, v_0) = \frac{p}{|||p|||} = \sqrt{|K|}\, p.$$

Remark 4.27. Recall that given the Levi-Civita connection $\nabla : TM^K \times TM^K \to TM^K$ in M^K, the *Christoffel symbols* are defined as the functions Γ_{ij}^k such that

$$\nabla_{\mathbf{E}_i} \mathbf{E}_j = \Gamma_{ij}^1\, \mathbf{E}_1 + \Gamma_{ij}^2\, \mathbf{E}_2,$$

where $\{\mathbf{E}_1, \mathbf{E}_2\}$ is a local frame for TM^K on an open subset of M^K.

Thanks to the normal vector of M^K, an extrinsic point of view is also possible. Let $\mathbf{x}$ be a chart of M^K. The normal vector $\mathbf{N}^{\mathbf{x}}$ to M^K is such that

$$\mathcal{B}\big(\mathbf{N}^{\mathbf{x}}(u,v),\, \mathbf{x}_u(u,v)\big) = \mathcal{B}\big(\mathbf{N}^{\mathbf{x}}(u,v),\, \mathbf{x}_u(u,v)\big) = 0.$$

On the one hand, since

$$\big\|\mathbf{N}^{\mathbf{x}}(u,v)\big\|^2 = \operatorname{sgn}(K),$$

we have that $\mathbf{N}^{\mathbf{x}}(u,v)$ is time-like if $K < 0$. On the other hand, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are space-like and they form a basis of the tangent plane T_pM at $p = \mathbf{x}(u,v)$. Therefore, the set

$$\big\{\mathbf{N}^{\mathbf{x}}(u,v),\, \mathbf{x}_u(u,v),\, \mathbf{x}_v(u,v)\big\}$$

is basis of $\mathbb{R}^3$ and we have:

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma_{11}^1\,\mathbf{x}_u + \Gamma_{11}^2\,\mathbf{x}_v + \mathcal{B}\big(\mathbf{x}_{uu}, \mathbf{N}^{\mathbf{x}}\big)\,\mathbf{N}^{\mathbf{x}},\\
\mathbf{x}_{uv} &= \Gamma_{12}^1\,\mathbf{x}_u + \Gamma_{12}^2\,\mathbf{x}_v + \mathcal{B}\big(\mathbf{x}_{uv}, \mathbf{N}^{\mathbf{x}}\big)\,\mathbf{N}^{\mathbf{x}},\\
\mathbf{x}_{vv} &= \Gamma_{22}^1\,\mathbf{x}_u + \Gamma_{22}^2\,\mathbf{x}_v + \mathcal{B}\big(\mathbf{x}_{vv}, \mathbf{N}^{\mathbf{x}}\big)\,\mathbf{N}^{\mathbf{x}}.
\end{aligned}$$

The relations above constitute the common definition in classical differential geometry of the Christoffel symbols Γ_{ij}^k. Notice that the connection ∇ is such that

$$\nabla_{\mathbf{x}_i}\mathbf{x}_j = \mathbf{x}_{ij} - \mathcal{B}\big(\mathbf{x}_{ij},\, \mathbf{N}^{\mathbf{x}}\big)\,\mathbf{N}^{\mathbf{x}}.$$

Remark 4.28. Note that the normalization in the definition of the normal vector is done with $|||\cdot|||$ rather than $\|\cdot\|$. In the case $K > 0$ both normalizations would be equivalent, but in the hyperbolic case ($K < 0$) a normalization by $\|\cdot\|$ would lead to a normal vector having a purely imaginary number in all its coordinates. It is the case, for instance, of the normal vector

$$\tilde{\mathbf{N}}^{\mathbf{x}}(u,v) = \frac{(\mathbf{x}_u \wedge \mathbf{x}_v)(u,v)}{\big\|(\mathbf{x}_u \wedge \mathbf{x}_v)(u,v)\big\|} = \sqrt{K}\,\mathbf{x}(u,v).$$

This normal vector verifies $\big\|\tilde{\mathbf{N}}^{\mathbf{x}}\big\| = 1$. In the case $K < 0$, it is related to the usual definition of the the *Gauss map* as an application from M^K into the *de Sitter space*

$$\mathbb{S}_1^2 = \big\{\mathbf{x} \in \mathbb{R}^{1,2} : \mathcal{B}(\mathbf{x},\mathbf{x}) = 1\big\}.$$

Nevertheless, if complex numbers are wanted to be avoided, Definition 4.24 is more appropriate.

The relation between $\mathbf{N}^{\mathbf{x}}$ and $\tilde{\mathbf{N}}^{\mathbf{x}}$ is

$$\mathbf{N}^{\mathbf{x}} = \sqrt{\operatorname{sgn}(K)}\,\tilde{\mathbf{N}}^{\mathbf{x}}.$$

As a comment, if we remove the sign of K taken in Definition 4.24, then the expression for the new normal vector would be equal to

$$\pm\,\mathrm{sgn}(K)\,\sqrt{|K|}\,\mathbf{x}(u,v)$$

instead of the one given in Proposition 4.25. To avoid this sign of K here, we take it in the definition of the normal vector from the beginning.

Once a normal vector has been defined in M^K, it induces an orientation in M^K. Henceforth, this (positive) orientation will be considered in M^K.

Orthonormal frame adapted to a curve in M^K

Let $\alpha : I \to M^K$ be an arc-length parameterized regular curve, with I being some interval. The normal vector to M^K at a point $\alpha(s) = \mathbf{x}\big(u(s), v(s)\big)$ is

$$\mathbf{N}(s) := \mathbf{N}^{\mathbf{x}}\big(u(s), v(s)\big) = \sqrt{|K|}\,\alpha(s).$$

With that, an orthonormal frame adapted to α is naturally defined:

$$\begin{aligned}\mathcal{F}(s) &= \big\{\mathbf{N}(s),\, \mathbf{t}(s),\, (\mathbf{N}\wedge\mathbf{t})(s)\big\}\\ &= \big\{\sqrt{|K|}\,\alpha(s),\, \alpha'(s),\, \sqrt{|K|}\,(\alpha\wedge\alpha')(s)\big\}.\end{aligned}$$

This frame is called the *Darboux frame of α on M^K* (it is also called the *Sabban frame* in the spherical case).

It is straightforward to show that the variation of $\mathcal{F}$ is given by

$$\begin{aligned}&\begin{pmatrix}\sqrt{|K|}\,\alpha(s)\\ \alpha'(s)\\ \sqrt{|K|}\,(\alpha\wedge\alpha')(s)\end{pmatrix}'\\ &\qquad= \begin{pmatrix}0 & \sqrt{|K|} & 0\\ -\,\mathrm{sgn}(K)\sqrt{|K|} & 0 & \kappa_g^{\alpha}(s)\\ 0 & -\kappa_g^{\alpha}(s) & 0\end{pmatrix}\begin{pmatrix}\sqrt{|K|}\,\alpha(s)\\ \alpha'(s)\\ \sqrt{|K|}\,(\alpha\wedge\alpha')(s)\end{pmatrix}.\end{aligned}$$

Therefore, if $\big(x(s),\, y(s),\, z(s)\big)$ are the coordinates of a curve on M^K in the frame $\mathcal{F}$, then we can compute the coordinates on $\mathcal{F}$ of the derivative of such a curve just by

$$\begin{aligned}&\big(x'(s) \quad y'(s) \quad z'(s)\big)\\ &\qquad+ \big(x(s) \quad y(s) \quad z(s)\big)\begin{pmatrix}0 & \sqrt{|K|} & 0\\ -\,\mathrm{sgn}(K)\sqrt{|K|} & 0 & \kappa_g^{\alpha}(s)\\ 0 & -\kappa_g^{\alpha}(s) & 0\end{pmatrix}.\end{aligned}$$

Angles in M^K

The definition of the angle between two vectors of M^K depends on the geometry of the space. The points of the manifold M^K, seen as vectors in $\mathbb{R}^3$, are space-like if $K > 0$ but time-like if $K < 0$. Therefore, the definition of an angle is different according to the sign of K.

Let p and q be two points of M^K. If $K > 0$, the angle between p and q is defined as the *Euclidean angle between those vectors*, i.e., as the unique value $\theta(p,q) \in [0,\pi]$ such that

$$\mathcal{B}(p,q) = \|p\| \, \|q\| \cos\theta(p,q) = \frac{\cos\theta(p,q)}{K}. \tag{4.5}$$

If $K < 0$, the vectors p and q are time-like. There is a unique non-negative real number $\theta(p,q)$ such that

$$\mathcal{B}(p,q) = \|p\| \, \|q\| \cosh\theta(p,q) = \frac{\cosh\theta(p,q)}{K}. \tag{4.6}$$

The value $\theta(p,q)$ is called the *Lorentzian time-like angle between p and q.*

The expressions (4.5) and (4.6) can be compacted in a single equation:

$$\mathcal{B}(p,q) = \|p\| \, \|q\| \cos\psi(p,q) = \frac{\cos\psi(p,q)}{K}, \tag{4.7}$$

where $\psi(p,q) = \sqrt{\operatorname{sgn}(K)}\,\theta(p,q)$, with $\theta(p,q)$ being the Euclidean angle (resp. Lorentzian time-like angle) between p and q if $K > 0$ (resp. $K < 0$).

With that, define the *angle between p and q in M^K* as the value

$$\theta(p,q) = \frac{1}{\sqrt{\operatorname{sgn}(K)}} \arccos\big(K\,\mathcal{B}(p,q)\big).$$

Some authors (as Santaló in [83]) define the angle in the non-Euclidean space by the value $\psi(p,q)$, which is a non-negative purely imaginary number in the hyperbolic case.

Geodesic curvature

As happens in classical differential geometry of surfaces in $\mathbb{R}^3$, it is easy to show that the covariant derivative can be computed using the defined normal vector (see Remark 4.27).

Proposition 4.29. *Let $\mathbf{v} : I \to TM^K$ be a tangent vector field along a differentiable curve α. The covariant derivative of $\mathbf{v}$ along α verifies*

$$D_t\mathbf{v}(t) = \mathbf{v}'(t) - \mathcal{B}\big(\mathbf{v}'(t), \mathbf{N}(t)\big)\,\mathbf{N}(t),$$

where $\mathbf{N}(t)$ is the normal vector to M^K at the point $\alpha(t)$.

Let $\alpha : I \to M^K$ be an arc-length parameterized regular curve, where I is some interval. The *(unsigned) geodesic curvature of* α is defined as the function $\kappa_g^\alpha : I \to \mathbb{R}$ given by

$$\kappa_g^\alpha(s) = \big\|D_t\alpha'(s)\big\|.$$

Since $\alpha''(s)$ is orthogonal to $\alpha'(s)$, it can be written in the frame $\mathcal{F}$ as follows:

$$\alpha''(s) = \mathbf{t}'(s) = \mathcal{B}\big(\mathbf{t}'(s), \mathbf{N}(s)\big)\,\mathbf{N}(s) + \mathcal{B}\big(\mathbf{t}'(s), (\mathbf{N}\wedge\mathbf{t})(s)\big)\,(\mathbf{N}\wedge\mathbf{t})(s).$$

From this, using Proposition 4.29, we have that

$$\kappa_g^\alpha = \big|\mathcal{B}(\alpha'', \alpha\wedge\alpha')\big|.$$

With the same idea, the *(signed) geodesic curvature of* α is defined by

$$\kappa_g^\alpha := \mathcal{B}\big(\mathbf{t}',\, \mathbf{N}\wedge\mathbf{t}\big) = \sqrt{|K|}\,\mathcal{B}\big(\alpha'',\, \alpha\wedge\alpha'\big).$$

From now on, the geodesic curvature to be considered will be the signed one.

In general, if a curve γ is not parameterized by arc length, the geodesic curvature of γ can be computed with

$$\kappa_g^\gamma(t) = \sqrt{|K|}\,\frac{\det\big(\gamma(t), \gamma'(t), \gamma''(t)\big)}{\|\gamma'(t)\|^3}. \tag{4.8}$$

Geodesics

First, the definition of straight lines in M^K is given and, later, it is seen that they correspond with geodesics in M^K

Definition 4.30 (Straight line in M^K)**.** A *(non-Euclidean) straight line of* M^K is defined as the intersection of M^K with a 2-dimensional vector subspace of $\mathbb{R}^3$ that passes through the origin.

Proposition 4.31. *The unit speed straight line γ in M^K that passes through $p \in M^K$ at $t = 0$ in the direction of a unit vector $\mathbf{v} \in T_pM^K$ can be parameterized by*

$$\gamma(t) = \cos\big(\sqrt{K}\,t\big)\,p + \frac{\sin\big(\sqrt{K}\,t\big)}{\sqrt{K}}\,\mathbf{v}.$$

Proof. For any $p \in M^K$ and a unit tangent vector $\mathbf{v} \in T_pM^K$, we have that $\{p, \mathbf{v}\}$ is an orthogonal basis for the 2-dimensional subspace that pass through the origin, p and contains the vector $\mathbf{v}$. Therefore, it will define a straight line in M^K and any point $\gamma(t)$ of this line will be spanned by those two vectors:

$$\gamma(t) = \lambda(t)\,p + \mu(t)\,\mathbf{v}. \tag{4.9}$$

Multiplying (4.9) by p with the bilinear form $\mathcal{B}$, we get

$$\mathcal{B}\big(\gamma(t), p\big) = \frac{\lambda(t)}{K},$$

which, since $\gamma(t) \in M^K$, it is equivalent to $\lambda(t) = \cos\psi\big(\gamma(t), p\big)$. Also, we have that

$$\frac{1}{K} = \mathcal{B}\big(\gamma(t), \gamma(t)\big) = \frac{\lambda^2(t)}{K} + \mu^2(t).$$

This means that $1 = \lambda^2(t) + K\,\mu^2(t)$. Therefore,

$$\mu^2(t) = \frac{1 - \lambda^2(t)}{K} = \frac{\sin^2\psi\big(\gamma(t), p\big)}{K}$$

and, from (4.9), we have:

$$\gamma(t) = \cos\psi\big(\gamma(t), p\big)\, p + \frac{\sin\psi\big(\gamma(t), p\big)}{\sqrt{K}}\,\mathbf{v}.$$

Denote $\bar{\psi}(t) := \psi\big(\gamma(t), p\big)$. The derivative of γ is

$$\gamma'(t) = -\bar{\psi}'(t)\sin\bar{\psi}(t)\, p + \frac{\bar{\psi}'(t)}{\sqrt{K}}\cos\bar{\psi}(t)\,\mathbf{v}.$$

Now, since

$$1 = \mathcal{B}\big(\gamma'(t), \gamma'(t)\big) = \frac{\bar{\psi}'^2(t)}{K},$$

we deduce $\bar{\psi}(t) = \sqrt{K}\,t + C$, with C being a real constant. The condition $\gamma(0) = p$ yields $C = 0$. Thus, the equation of the statement is found. □

Recall that α is a *geodesic in M^K with respect to ∇* if $D_t\alpha' = 0$, i.e., if $\kappa_g^\alpha = 0$. As a consequence of the definition, next result arises.

Proposition 4.32. *Let $\alpha : I \to M^K$ be an arc-length parameterized curve with non-vanishing curvature κ. The curve α is a geodesic if and only if the normal vector $\mathbf{n}$ of α is parallel to the normal vector $\mathbf{N}$ of M^K at each point.*

Proof. The curve α is a geodesic if $D_t\alpha' = 0$. Since

$$\begin{aligned} D_t\alpha' &= \alpha'' - \mathcal{B}\big(\alpha'', \mathbf{N}\big)\,\mathbf{N} = \mathbf{t}' - \mathcal{B}\big(\mathbf{t}', \mathbf{N}\big)\,\mathbf{N} \\ &= \kappa\,\mathbf{n} - \kappa\,\mathcal{B}(\mathbf{n}, \mathbf{N})\,\mathbf{N}, \end{aligned}$$

immediately we have that α is geodesic if and only if $\mathbf{n}$ is parallel to $\mathbf{N}$. □

With that, we can easily check that the straight lines of M^K are geodesics.

Proposition 4.33. *The unit speed straight line that passes through $p \in M^K$ at $t = 0$ in the direction of a unit vector $\mathbf{v} \in T_pM^K$,*

$$\gamma(t) = \cos\big(\sqrt{K}\,t\big)\, p + \frac{\sin\big(\sqrt{K}\,t\big)}{\sqrt{K}}\,\mathbf{v},$$

is a geodesic in M^K.

Proof. We have that

$$\gamma'(t) = -\sqrt{K}\sin\bigl(\sqrt{K}\,t\bigr)\,p + \cos\bigl(\sqrt{K}\,t\bigr)\,\mathbf{v},$$
$$\gamma''(t) = -K\cos\bigl(\sqrt{K}\,t\bigr)\,p - \sqrt{K}\sin\bigl(\sqrt{K}\,t\bigr)\,\mathbf{v}.$$

Thus,

$$\mathcal{B}\bigl(\gamma''(t),\gamma(t)\bigr) = -K\cos^2\bigl(\sqrt{K}\,t\bigr)\,\mathcal{B}(p,p) - \sin^2\bigl(\sqrt{K}\,t\bigr)\,\mathcal{B}(\mathbf{v},\mathbf{v}) = -1,$$

which means that the normal vector of γ is parallel to γ and, thus, to the normal vector of M^K at each point of γ. $\square$

Definition 4.34 (Chord in M^K)**.** Define the *chord between* p *and* q *on* M^K as the shortest piece of the geodesic curve that passes through p and q and has these points as endpoints.

Distance map

In any connected Riemannian manifold M, given $p, q \in M$, a distance $\mathrm{d}(p,q)$ can be defined as the infimum of the lengths of all piecewise regular curves from p to q.

Thus, a metric space structure can be outfitted to M^K: given two points $p, q \in M^K$, the distance between them is defined by the length of the chord between p and q.

Let γ be the minimizing unit speed geodesic such that $\gamma(0) = p$ and $\gamma(t_0) = q$. Since γ is arc-length parameterized, the distance from p to q will be equal to t_0. On the one hand, by Proposition 4.33, unit speed straight lines are geodesics. On the other hand, we have seen in the proof of Proposition 4.31 that

$$\bar{\psi}(t) = \psi\bigl(\gamma(t),p\bigr) = \sqrt{K}\,t.$$

Therefore,

$$\bar{\psi}(t_0) = \psi(q,p) = \sqrt{K}\,t_0.$$

From this follows the next definition.

Definition 4.35 (Distance function in M^K)**.** The *distance between* p *and* q *in* M^K is defined by

$$\mathrm{d}(p,q) := \frac{\psi(p,q)}{\sqrt{K}} = \frac{1}{\sqrt{K}}\,\arccos\bigl(K\,\mathcal{B}(p,q)\bigr). \tag{4.10}$$

Proposition 4.36. *The direction of the unit speed geodesic* γ *that passes through* p *and* q *such that* $\gamma(0) = p$ *is*

$$\mathbf{v} = -\frac{p\wedge(p\wedge q)}{\bigl\|p\wedge(p\wedge q)\bigr\|} = \frac{\sqrt{K}}{\sin\bigl(\sqrt{K}\,\mathrm{d}(p,q)\bigr)}\Bigl(q - \cos\bigl(\sqrt{K}\,\mathrm{d}(p,q)\bigr)\,p\Bigr).$$

Proof. We must find a direction $\mathbf{v}$ such that

$$q = \cos\bigl(\sqrt{K}\,\mathrm{d}(p,q)\bigr)\,p + \frac{\sin\bigl(\sqrt{K}\,\mathrm{d}(p,q)\bigr)}{\sqrt{K}}\,\mathbf{v}.$$

From this equation, it is deduced that

$$\mathbf{v} = \frac{\sqrt{K}}{\sin\bigl(\sqrt{K}\,\mathrm{d}(p,q)\bigr)}\Bigl(q - \cos\bigl(\sqrt{K}\,\mathrm{d}(p,q)\bigr)\,p\Bigr).$$

Now it is enough to see that the second equality of the statement holds. With Proposition 4.16 and the definition of $\psi(p,q)$ in Equation (4.7), we have

$$p \wedge (p \wedge q) = \frac{1}{|K|}\bigl(\cos\psi(p,q)\,p - q\bigr).$$

Also,

$$\bigl\|p \wedge (p \wedge q)\bigr\|^2 = \frac{\sin^2\psi(p,q)}{K^3}.$$

Thus,

$$\frac{p \wedge (p \wedge q)}{\bigl\|p \wedge (p \wedge q)\bigr\|} = \frac{\sqrt{K}}{\sin\psi(p,q)}\Bigl(\cos\psi(p,q)\,p - q\Bigr).$$

Since $\psi(p,q) = \sqrt{K}\,\mathrm{d}(p,q)$, the proof is finished. $\square$

Note that in the hyperbolic case ($K < 0$), since

$$-\mathbf{i}\,\arccos(x) = \operatorname{argcosh}(x), \quad \text{for all } x \geq 1,$$

the expression for the distance map between two points $p, q \in \mathbb{H}^2(K)$ can be written as

$$\mathrm{d}(p,q) = \frac{1}{\sqrt{-K}}\,\operatorname{argcosh}\bigl(K\,\mathcal{B}(p,q)\bigr).$$

Recall that the domain of argcosh is $[1, +\infty[$. The previous identity makes sense because of the following result.

Proposition 4.37. *If $p, q \in \mathbb{H}^2(K)$, then $\mathcal{B}(p,q) \leq \frac{1}{K}$. Equality holds if $p = q$.*

Proof. The hyperboloid $\mathbb{H}^2(K)$ has curvature K. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathcal{B}(p,q) &= -p_1\,q_1 + p_2\,q_2 + p_3\,q_3 = -p_1\,q_1 + \bigl\langle (p_2,p_3),(q_2,q_3)\bigr\rangle \\ &\leq -p_1\,q_1 + \bigl\|(p_2,p_3)\bigr\|\,\bigl\|(q_2,q_3)\bigr\|.\end{aligned}$$

Since $p, q \in \mathbb{H}^2(K)$, we have that $-p_1\,q_1 + \bigl\|(p_2,p_3)\bigr\|\,\bigl\|(q_2,q_3)\bigr\| \leq \frac{1}{K}$ is equivalent to

$$\sqrt{p_1^2 + \frac{1}{K}}\,\sqrt{q_1^2 + \frac{1}{K}} \leq p_1 q_1 + \frac{1}{K}.$$

Note that $p_1^2 + \frac{1}{K} \geq 0$ and $q_1^2 + \frac{1}{K} \geq 0$. Thus, it is enough to see if

$$\left(p_1^2 + \frac{1}{K}\right)\left(q_1^2 + \frac{1}{K}\right) \leq \left(p_1 q_1 + \frac{1}{K}\right)^2.$$

This expression can be written as

$$\frac{1}{K}\left(p_1^2 + q_1^2\right) \leq \frac{2}{K}\, p_1 q_1,$$

that is to say,

$$\frac{1}{K}(p_1 - q_1)^2 \leq 0.$$

This last inequality holds for $K < 0$, which completes the proof. □

Angles between curves

Given two space-like vectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^3$ that span a space-like vector subspace, the *space-like angle between* $\mathbf{x}$ *and* $\mathbf{y}$ is defined as the unique real number $\theta(\mathbf{x}, \mathbf{y}) \in [0, \pi]$, such that

$$\mathcal{B}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos\theta(\mathbf{x}, \mathbf{y}).$$

For $K > 0$, it is the common definition of the *Euclidean angle between two vectors of* $\mathbb{R}^3$. If $K < 0$, it is the *Lorentzian space-like angle between two space-like vectors of* $\mathbb{R}^{1,2}$.

Let $s_0 \in I$ and $\alpha, \beta : I \to M^K$ be two geodesics in M^K such that $\alpha(s_0) = \beta(s_0)$. The vectors $\alpha'(s_0)$ and $\beta'(s_0)$ are space-like vectors, so they span a space-like vector subspace of $\mathbb{R}^3$—the tangent plane $T_{\alpha(s_0)}M^K$.

The *angle between* α *and* β is defined as the space-like angle between $\alpha'(s_0)$ and $\beta'(s_0)$.

Areas

Finally, given a compact region R of M^K (defined by the boundary of some simple closed curve ϕ in M^K) contained in some coordinate neighborhood $\mathbf{x}(U)$ consistent with the orientation of M^K, the *area of* R is defined, as it is usual in Riemannian manifolds (see [21]), by

$$\mathcal{A}(\phi) = \iint_{\mathbf{x}^{-1}(R)} \sqrt{(g_{11}\, g_{22} - g_{12}^2)(u, v)}\, \mathrm{d}u\, \mathrm{d}v,$$

where g_{ij} is the local representation of the Riemannian metric g in the coordinate system $\mathbf{x}$. Also, by using Proposition 4.16, it is easy to show that

$$\left\|(\mathbf{x}_u \wedge \mathbf{x}_v)(u, v)\right\|^2 = \operatorname{sgn}(K)\,(g_{11}\, g_{22} - g_{12}^2)(u, v).$$

The expression

$$\mathrm{d}A := \sqrt{(g_{11}\, g_{22} - g_{12}^2)(u,v)}\, \mathrm{d}u\, \mathrm{d}v \geq 0,$$

is called the *area element*. Notice that $\mathcal{A}(\phi) \geq 0$. In the same way as in the plane, the signed area $\tilde{\mathcal{A}}(\phi)$ encircled by ϕ is defined to be $\mathcal{A}(\phi)$ if ϕ is positively oriented or $-\mathcal{A}(\phi)$ otherwise.

4.3 Some kinds of curves in M^K

In this section, a brief introduction to the non-Euclidean versions of the curves studied in Chapter 1 will be given. The extended results to those given there on areas and lengths will be deduced in next chapter.

4.3.1 Parallel curves in M^K

As seen in the previous section, an orthonormal basis of the tangent plane $T_{\alpha(s)}M^K$, which is a space-like subspace of $\mathbb{R}^3$, is given by the vectors $\mathbf{t}(s)$ and $\sqrt{|K|}\,(\alpha \wedge \mathbf{t})(s)$. The second vector is the *intrinsic normal vector of the curve α at $\alpha(s)$*.

Definition 4.38 (Parallel curve in M^K)**.** Let $\alpha : I \to M^K$ be a regular parametric curve (positively oriented in M^K) and $d > 0$. The *(inner) parallel curve to α at a distance d in M^K* is defined as the curve $\alpha_d : I \to M^K$ given by

$$\alpha_d(s) = \cos\big(\sqrt{K}\,d\big)\,\alpha(s) + \frac{\sin\big(\sqrt{K}\,d\big)}{\sqrt{K}}\,\sqrt{|K|}\,(\alpha \wedge \mathbf{t})(s).$$

If the sign $+$ is replaced by $-$, it is the definition for the *outer parallel curve to α at a distance d in M^K*.

The notion of a parallel curve to another according to a constant angle given in Section 1.2.2 can also be generalized. Very few references can be found in the literature where such an extension is generalized to constant curvature surfaces, [94], or even to any surface, [92]. Those two cited articles were written by Vidal Abascal in Spanish and published more than 70 years ago.

Definition 4.39 (Parallel curve in M^K according to an angle ω)**.** Let $\alpha : I \to M^K$ be a regular parametric curve, $d > 0$ and $\omega \in \,]-\pi, \pi]$. The *parallel curve to α at a distance d in M^K according to a constant angle ω* is defined as the curve $\alpha_{d,\omega} : I \to M^K$ given by

$$\alpha_{d,\omega}(s) = \cos\big(\sqrt{K}\,d\big)\,\alpha(s) + \frac{\sin\big(\sqrt{K}\,d\big)}{\sqrt{K}}\,\big(\cos\omega\,\mathbf{t}(s) + \sin\omega\,\sqrt{|K|}\,(\alpha \wedge \mathbf{t})(s)\big).$$

See in Figure 4.1 an example of an orthogonal parallel curve and a parallel curve according to an angle $\omega \neq \pi/2$.

Note that Definitions 4.38 and 4.39 use the parametric representation. In the same way as in the plane (see Section 1.2), other definitions of the parallel curves might also be possible in non-Euclidean geometry.

Moreover, analogous results to Steiner's formulae of the plane can be given in 2-dimensional constant curvature manifolds. In the next chapter, these expressions will be deduced from a more general setting.

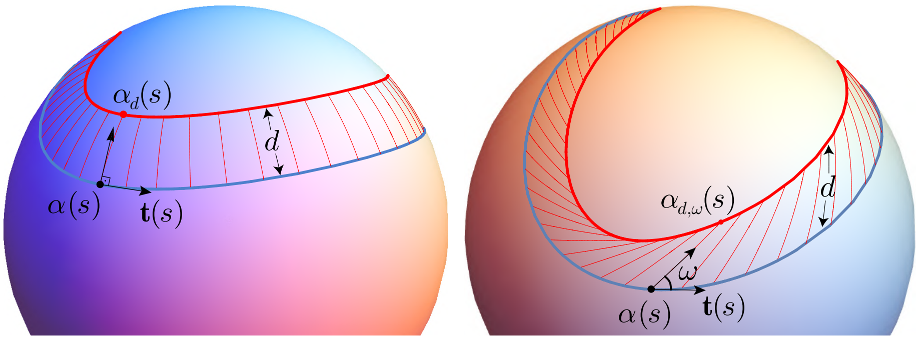

Figure 4.1: On the left, an example of a parallel curve in M^K at a distance d. On the right, the same but according to a constant non-right angle ω.

4.3.2 Constant width curves in M^K

In this section, the definition of a width in M^K together with the generalized notion of *constant width curves* will be given. Before that, the definition of a convex curve in M^K shall be introduced (see [80] and [82]).

Definition 4.40 (Convex curve in M^K)**.** A closed curve in M^K is said to be *convex* if it cannot be cut by any geodesic in more than two points, except that a complete arc of geodesic may belong to the curve.

This definition of convexity can be related to the notion of supporting geodesic. The immediate generalization of supporting lines to non-Euclidean geometry consists in taking geodesic lines instead. The main difference from the plane is regarding the parallel postulate. For instance, two geodesics in the sphere will intersect to each other.

The definition given below corresponds to the approach of Santaló (see [80], [82] and [81]).

Definition 4.41 (Supporting geodesic in M^K)**.** Given a closed curve α in M^K, any geodesic with only one point in common with α or with a complete geodesic arc in common is called a *supporting geodesic of* α.

As happens in the plane, regular convex curves in M^K which determine a finite region verify that their tangent geodesics are supporting geodesics.

The next result is sometimes taken as the definition of a convex curve in M^K. No good single reference has been found for it; for the case $\mathbb{S}^2$, see [60] and for $\mathbb{H}^2$, see [32] and [84].[2]

Theorem 4.42. *A $\mathcal{C}^2$-closed simple curve in M^K is convex if and only if its geodesic curvature does not change its sign.*

[2]Acknowledgements to Paul Bryan, Olga Gil, Vicente Miquel and Agustí Reventós for their interest in this problem.

Some definitions of *width* for non-Euclidean curves can be found in the literature. The notion of constant width curve follows directly once the width has been defined. In the paper [56] of Leichtweiss, an interesting introduction to non-Euclidean constant width curves is given.

The first definition of constant width curve in non-Euclidean geometry is due to Blaschke, who introduced the concept for the sphere as follows: a convex curve in the sphere is called of constant width ℓ if it coincides with its inner parallel curve at a distance ℓ.

The next definition of a width corresponds with the approach of Santaló (visualize it in Figure 4.2).

Definition 4.43 (Width of a curve in M^K)**.** Let α be a closed curve in M^K. Let g_1 be a supporting geodesic of α and let A_1 be a common point of g_1 and α. Consider the orthogonal geodesic, say g, to g_1 at A_1. Now, there exists another supporting geodesic of α, g_2, which is orthogonal to g at a point A_2 and has a common point with α. The *width of α corresponding to the point* A_1 is defined to be the length of the geodesic arc A_1A_2 of g.

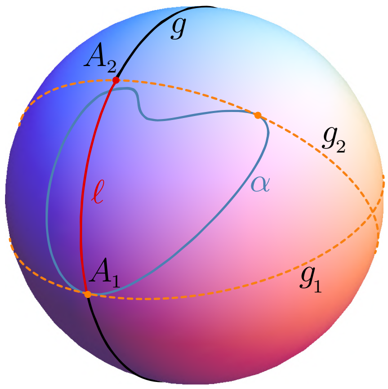

Figure 4.2: Definition of the width of a curve in M^K corresponding to a point A_1 defined by a supporting geodesic.

In the spherical case, the distance between A_1 or A_2 and any of the two intersections of g_1 with g_2 is equal to $\pi/2$. Thus, the width of a curve in the sphere can be measured with the angle between two supporting geodesics.

Definition 4.44 (Constant width curve in M^K)**.** A convex curve in M^K is called a *curve of constant width* if its width is the same for any supporting geodesic to any point.

In [56] (see also [55]) another notion of constant width curves is given by using support functions in the non-Euclidean space, which is proven to coincide with Blaschke's definition of self-parallel curves and with Santaló's definition. As in the plane (see Section 1.3.2), support functions allow one to easily parameterize constant width curves in M^K. The interested reader can look at the cited articles. In particular, the analogue of Proposition 1.17 in the plane is true.

Proposition 4.45. *In a curve of constant width in M^K, each geodesic chord of length the constant width joining the contact points of the corresponding supporting geodesics is orthogonal to those geodesics.*

See in Figure 4.3 an example of a constant width curve constructed by the approach of support functions.

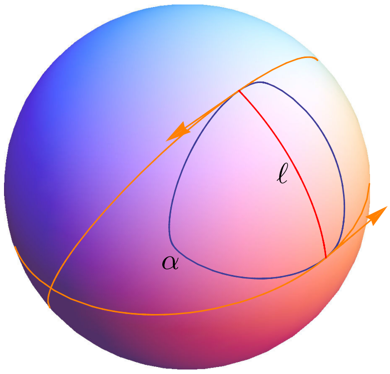

Figure 4.3: Example of a curve α of constant width ℓ in M^K.

As in the plane, there exists a version of Barbier's theorem to constant curvature surfaces, which was given by Vidal Abascal in [91]. In Section 5.3.4, Theorem 5.22, this result will be stated and proved in a different way with a new approach.

4.3.3 Bicycle curves in M^K

The notion of a bicycle tire-track curve seen in the plane can be extended to Riemannian manifolds (see e.g. [61] and [9]). In this case, the bicycle is represented by a moving geodesic chord of constant length. The constraint imposed on the chord is to be always in the tangent direction to the path of the rear wheel curve (see Figure 4.4).

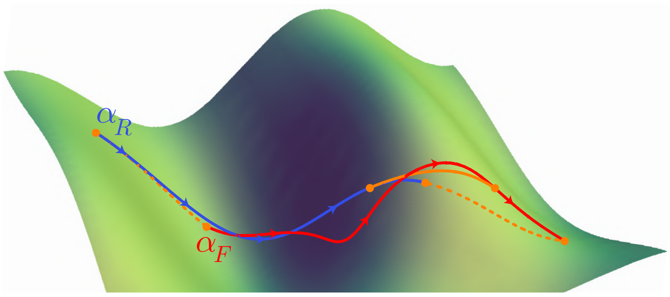

Figure 4.4: The bicycle is represented by a geodesic chord of constant length always in the tangent direction of the rear tire track α_R. The front wheel is the curve α_F.

In the constant curvature manifold M^K, if α_R and α_F represent the rear and the front wheel, respectively, of a bicycle of length ℓ, then

$$\alpha_F(t) = \cos\bigl(\sqrt{K}\,\ell\bigr)\,\alpha_R(t) + \frac{\sin\bigl(\sqrt{K}\,\ell\bigr)}{\sqrt{K}}\,\mathbf{t}_R(t),$$

where $\mathbf{t}_R$ represents the tangent vector of the rear track α_R.

As in the plane, note that the front track α_F is defined as a parallel curve to α_R at a distance ℓ according to a constant angle $\omega = 0$. In Section 5.3.4, a result on the area swept out by a closed tire-track movement will be given.

Chapter 5

Moving chords in constant curvature surfaces

The motion of a moving chord can be described by means of Jacobi fields along geodesics. It is an interesting tool for understanding the behavior of the movement (see [21] and [54] for an introduction to Jacobi fields). In this chapter, a brief introduction to the Jacobi fields induced by a moving chord both in the plane and in non-Euclidean geometry is given. The notion of *generated curve* is extended to constant curvature surfaces and the generalized version of Holditch's theorem, Steiner's formulae for parallel curves and Barbier's theorem are deduced from a new point of view. Finally, some results involving swept out areas and the closed curve which realizes the generalized Holditch constant are given.

5.1 Jacobi fields in the planar case

First of all, consider the planar case. Let $\alpha : I \to \mathbb{R}^2$ be an arc-length parameterized closed regular curve and let a moving chord of length ℓ move smoothly along the curve without retrograde motion. Assume α to be positively oriented. The moving chord can be parameterized for each point $s \in I$ as $\lambda_s : [0, \ell] \to \mathbb{R}^2$, $\lambda_s(u) = \alpha(s) + u\,\beta(s)$, where

$$\beta(s) = \cos\nu(s)\,\mathbf{t}(s) + \sin\nu(s)\,\mathbf{n}(s)$$

is its unit direction. The angle $\nu(s)$ is that defined in Section 3.1.3. The set $\big\{\lambda_s(u)\big\}_{s\in I}$ is a family of geodesics, so it induces naturally the Jacobi field

$$\mathcal{J}_s(u) = \frac{\mathrm{d}}{\mathrm{d}s}\lambda_s(u) = \mathbf{t}(s) + u\,\beta'(s).$$

Since $\lambda_s'(u) = \beta(s)$ and $\beta(s)$ is a unit vector,

$$\big\langle \mathcal{J}_s(u), \lambda_s'(u)\big\rangle = \big\langle \mathbf{t}(s), \beta(s)\big\rangle = \cos\nu(s).$$

Therefore, the Jacobi field $\mathcal{J}_s(u)$ can be split ([48], p. 68) into two components as

$$\mathcal{J}_s(u) = \cos\nu(s)\,\lambda_s'(u) + B_s(u), \tag{5.1}$$

with $B_s(u)$ being a normal Jacobi field along λ_s (see Figure 5.1).

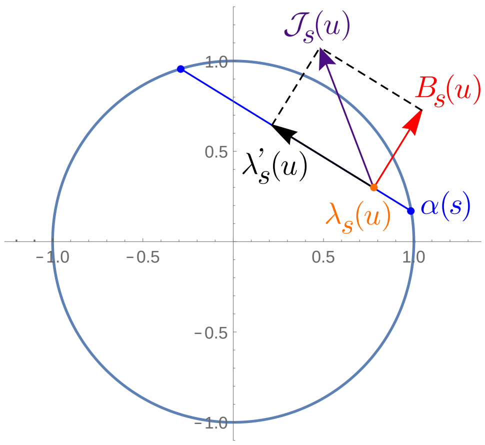

Figure 5.1: The Jacobi field $\mathcal{J}_s(u)$ is split into a tangential component, $\lambda_s'(u)$, and a normal one, $B_s(u)$.

Note that

$$\beta'(s) = -\big(\nu'(s) + \kappa(s)\big)\,\mathbf{v}(s),$$

where $\mathbf{v}(s) = \sin\nu(s)\,\mathbf{t}(s) - \cos\nu(s)\mathbf{n}(s)$. Hence, from Equation (5.1),

$$B_s(u) = \Big(\sin\nu(s) - u\,\big(\nu'(s) + \kappa(s)\big)\Big)\,\mathbf{v}(s).$$

The sign of the expression

$$\sin \nu(s) - u \left(\nu'(s) + \kappa(s)\right)$$

determines the direction of the normal Jacobi field and it may change from $u = 0$ to $u = \ell$ (see Figure 5.2).

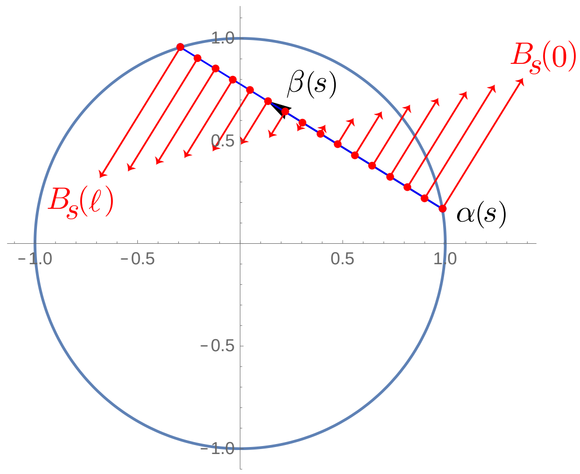

Figure 5.2: The vectors $B_s(u)$ given by the normal Jacobi field are shown for different $u \in [0, \ell]$ in the example of the circle. In the circle, the normal Jacobi field is zero in the midpoint of the rod.

The normal Jacobi field $B_s(u)$ is uniquely determined by the initial conditions:

$$B_s(0) = \sin \nu(s)\, \mathbf{v}(s) \quad \text{and} \quad D_u B_s(0) = B_s'(0) = -\left(\nu'(s) + \kappa(s)\right) \mathbf{v}(s).$$

5.2 Jacobi fields in non-zero constant curvature manifolds

Let M^K be the 2-dimensional Riemannian manifold of non-zero constant Gauss curvature K defined in Section 4.2. In this setting, a more general scenario is addressed: the movement of a geodesic chord of any (not necessarily constant) length sliding one of its extremities on α. Let $\beta(s)$ be the unit direction in $T_{\alpha(s)}M^K$ of such a geodesic chord from the point $\alpha(s)$. We summarize in the following hypothesis the conditions on α and β to be assumed.

Hypothesis $\mathcal{S}(\alpha)$. The initial curve $\alpha : I \to M^K$ is arc-length parameterized, positively oriented in M^K, differentiable, simple and regular, and for each point $\alpha(s)$ there is only one direction $\beta(s) \in T_{\alpha(s)}M^K$. Suppose that α has a finite length, so $I = [0, \mathcal{L}(\alpha)]$.

Suppose $\mathcal{S}(\alpha)$. The geodesic moving chord can be parameterized for each point $s \in I$ as the function $\lambda_s : [0, \ell] \to M^K$ defined by

$$\lambda_s(u) = \cos\bigl(\sqrt{K}\, u\bigr)\, \alpha(s) + \frac{\sin\bigl(\sqrt{K}\, u\bigr)}{\sqrt{K}}\, \beta(s),$$

where

$$\beta(s) = \cos\nu(s)\, \alpha'(s) + \sin\nu(s)\, \sqrt{|K|}\, (\alpha \wedge \alpha')(s) \tag{5.2}$$

is its unit direction in $T_{\alpha(s)}M^K$. The function $\nu(s) \in \mathbb{R}$ is the oriented angle function from $\alpha'(s)$ to the geodesic chord direction $\beta(s)$ according to the orientation of M^K given by its normal vector.

From now on, λ_s will denote this parameterized geodesic curve. The set $\bigl\{\lambda_s(u)\bigr\}_{s\in I}$ is a family of geodesics, so it induces naturally the Jacobi field

$$\mathcal{J}_s(u) = \frac{\mathrm{d}}{\mathrm{d}s}\lambda_s(u) = \cos\bigl(\sqrt{K}\, u\bigr)\, \alpha'(s) + \frac{\sin\bigl(\sqrt{K}\, u\bigr)}{\sqrt{K}}\, \beta'(s).$$

The orthogonality of α and β implies that

$$0 = \mathcal{B}\bigl(\alpha(s), \beta(s)\bigr)' = \mathcal{B}\bigl(\alpha'(s), \beta(s)\bigr) + \mathcal{B}\bigl(\alpha(s), \beta'(s)\bigr).$$

Since $\mathcal{B}\bigl(\alpha'(s), \beta(s)\bigr) = \cos\nu(s)$, we have $\mathcal{B}\bigl(\alpha(s), \beta'(s)\bigr) = -\cos\nu(s)$. With that, since

$$\lambda_s'(u) = -\sqrt{K}\sin\bigl(\sqrt{K}\, u\bigr)\, \alpha(s) + \cos\bigl(\sqrt{K}\, u\bigr)\, \beta(s),$$

we deduce

$$\mathcal{B}\bigl(\mathcal{J}_s(u),\, \lambda_s'(u)\bigr) = \cos\nu(s).$$

Hence, the Jacobi field $\mathcal{J}_s(u)$ can be split ([48], p. 68) into two components as

$$\mathcal{J}_s(u) = \cos\nu(s)\, \lambda_s'(u) + B_s(u), \tag{5.3}$$

with $B_s(u)$ being a normal Jacobi field along λ_s. Now, if

$$\mathbf{v}(s) := \sin\nu(s)\,\alpha'(s) - \cos\nu(s)\,\sqrt{|K|}\,(\alpha\wedge\alpha')(s), \tag{5.4}$$

then, using the variation of the frame $\mathcal{F}$,

$$\beta'(s) = -K\cos\nu(s)\,\alpha(s) - \big(\nu'(s)+\kappa_g(s)\big)\,\mathbf{v}(s).$$

Thus,

$$\begin{aligned}\mathcal{J}_s(u) &= \cos\big(\sqrt{K}\,u\big)\,\alpha'(s)\\ &\quad+\frac{\sin\big(\sqrt{K}\,u\big)}{\sqrt{K}}\Big(-K\cos\nu(s)\,\alpha(s)+\big(\nu'(s)+\kappa_g(s)\big)\,\mathbf{v}(s)\Big).\end{aligned}$$

Finally, the normal Jacobi field $B_s(u)$ along λ_s can be computed with (5.3) and written as $B_s(u) = b_s(u)\,\mathbf{v}(s)$, where

$$b_s(u) := \cos\big(\sqrt{K}\,u\big)\sin\nu(s) - \frac{\sin\big(\sqrt{K}\,u\big)}{\sqrt{K}}\,\big(\nu'(s)+\kappa_g(s)\big).$$

If $b_s(u) \geq 0$, then $b_s(u) = \big\|B_s(u)\big\|$. Nevertheless, in general, the sign of $b_s(u)$ may change from $u = 0$ to $u = \ell$ for a chord length ℓ.

Notice that $\mathbf{v}(s)$ is a unit parallel vector field along λ_s. Again, the uniqueness of the normal Jacobi field $B_s(u)$ is determined by the initial conditions:

$$B_s(0) = \sin\nu(s)\,\mathbf{v}(s) \quad\text{and}\quad D_uB_s(0) = B_s'(0) = -\big(\nu'(s)+\kappa_g(s)\big)\,\mathbf{v}(s).$$

Finally, recall that a Riemannian manifold with non-positive sectional curvature has no conjugate points. If the Riemannian manifold has constant positive sectional curvature K, the distance between two consecutive conjugate points is at most $\pi/\sqrt{K}$ (see [48], pp. 70, 74).

5.3 Generated curves in M^K

In this section, an extension of the results seen in Section 3.5 will be given, so that curves generated geodesically from another will be considered. The main contribution of this section is Lemma 5.12, which relates the geodesic curvatures of the generated curve and the initial one. As a consequence of this result, there follows an extension of Holditch's theorem in 2-dimensional constant curvature manifolds when two initial curves are considered instead of only one. By the same approach the 2-dimensional constant curvature version of Barbier's theorem for constant width curves and of Steiner's formulae for parallel curves can also be deduced.

Definition 5.1 (Generated curve in M^K)**.** Suppose that $\mathcal{S}(\alpha)$ holds. Given a continuous function $p : I \to [0, +\infty[$ (called the length function), generate from α the following curve:

$$\gamma(s) := \lambda_s\big(p(s)\big) = \cos\big(\sqrt{K}\, p(s)\big)\, \alpha(s) + \frac{\sin\big(\sqrt{K}\, p(s)\big)}{\sqrt{K}}\, \beta(s). \tag{5.5}$$

That is to say, for each $s \in I$, $\gamma(s)$ is generated by going from $\alpha(s)$ the length $p(s)$ geodesically in the direction forming an angle $\nu(s)$ with $\alpha'(s)$. The curve γ will be referred to as the *generated curve*. See in Figure 5.3 an example of a generated curve.

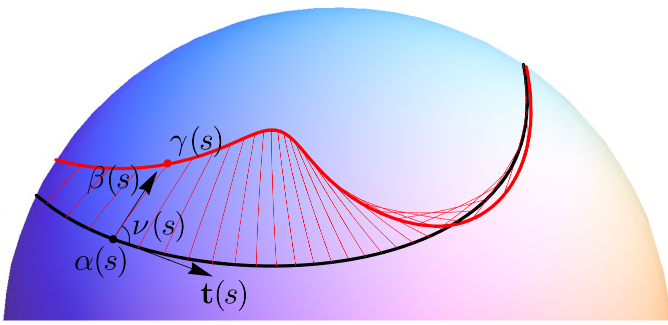

Figure 5.3: The initial curve α and its generated curve for a length function $p(s)$ and directions $\beta(s)$.

Notice that the uniqueness of $\beta(s)$ for each $\alpha(s)$ implies a non-retrograde movement of the endpoint lying in α in the generation of the new curve.

It is easy to show with a straightforward computation that

$$\left\|\gamma'(s)\right\|^2 = \big(p'(s) + \cos\nu(s)\big)^2 + \Big(b_s\big(p(s)\big)\Big)^2. \tag{5.6}$$

5.3.1 Regularity of generated curves

Notice that from Equation (5.6) the regularity of γ can be studied. In particular, the following hypothesis is equivalent to γ being regular.

Hypothesis $\mathcal{R}(\gamma)$. The length function $p(s)$ must satisfy that

$$p'(s) + \cos\nu(s) \neq 0$$

or

$$b_s\big(p(s)\big) = \cos\big(\sqrt{K}\,p(s)\big)\,\sin\nu(s) - \frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}}\,\big(\nu'(s) + \kappa_g^\alpha(s)\big) \neq 0$$

for each $s \in I$.

Now, it is convenient to find some sufficient condition for the hypothesis $\mathcal{R}(\gamma)$. First of all, to manage easily the function b_s, assume the following hypothesis.

Hypothesis $\mathcal{V}(\alpha)$. The hypothesis $\mathcal{S}(\alpha)$ holds and in addition

$$\nu'(s) + \kappa_g^\alpha(s) > 0$$

for all $s \in I$.

Remark 5.2. Both in spherical and hyperbolic geometry it is possible to interpret the geodesic curvature κ_g^α of a convex curve α by means of the derivative of some angle σ (as happens in the plane, see [80] and [82]). For each point of α, there is a tangent supporting geodesic. Given $s \in I$, the angle $\sigma(s)$ can be computed according to the metric as the space-like angle from the parallel transport of this tangent supporting geodesic along α and the corresponding supporting geodesic of α at $\alpha(s)$.

With that, the quantity

$$\nu'(s) + \kappa_g^\alpha(s) = \big(\nu(s) + \sigma(s)\big)'$$

can be understood as the derivative of the angle from such a parallel tangent vector field to the moving chord. Hypothesis $\mathcal{V}(\alpha)$ establishes this angle (for convex curves) to be an increasing function.

Proposition 5.3. *Suppose that $\mathcal{V}(\alpha)$ holds and, if $K > 0$, that $p(s) < \frac{\pi}{2\sqrt{K}}$ for all $s \in I$. If $\nu(s) < 0$ for all $s \in I$, then $\mathcal{R}(\gamma)$ holds. Otherwise, if in addition*

$$p(s) \neq \frac{1}{\sqrt{K}}\arctan\left(\frac{\sqrt{K}\,\sin\nu(s)}{\nu'(s) + \kappa_g^\alpha(s)}\right)$$

for all $s \in I$, then $\mathcal{R}(\gamma)$ holds.

Proof. First, notice that if $K < 0$, then

$$\cos\big(\sqrt{K}\,p(s)\big) = \cosh\big(\sqrt{-K}\,p(s)\big) \geq 0$$

and

$$\frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}} = \frac{\sinh\big(\sqrt{-K}\,p(s)\big)}{\sqrt{-K}} \geq 0.$$

In the case $K > 0$, since $p < \frac{\pi}{2\sqrt{K}}$, both terms are also non-negative.

If $\nu(s) < 0$ for all $s \in I$, then $b_s\big(p(s)\big) < 0$ and so $\mathcal{R}(\gamma)$ holds. Otherwise, the condition $b_s\big(p(s)\big) \neq 0$ can be written in the form of the statement as a sufficient condition for $\mathcal{R}(\gamma)$. □

In the particular case of being $p(s) = p$ constant (such as in the Holditch case), the condition of Proposition 5.3 is relaxed a little.

Theorem 5.4. *Suppose that $\mathcal{V}(\alpha)$ holds and $p(s) = p$ is constant. Suppose $p < \frac{\pi}{2\sqrt{K}}$ if $K > 0$. If $\nu(s) \neq \frac{\pi}{2}$ for all $s \in I$ for such a chord, then $\mathcal{R}(\gamma)$ holds. Otherwise, if $\nu(s)$ reaches the value $\frac{\pi}{2}$ at some point and*

$$p < \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\nu'_{\sup} + \kappa_g^{\sup}}\right), \tag{5.7}$$

or

$$p > \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\nu'_{\inf} + \kappa_g^{\inf}}\right), \tag{5.8}$$

(where the indices sup *and* inf *in a function denote the supremum or infimum, respectively, of such a function) then $\mathcal{R}(\gamma)$ also holds.*

Proof. From Equation (5.6), since $p(s) = p$ is constant, we have

$$\left\|\gamma'(s)\right\|^2 = \cos^2\nu(s) + b_s^2(p).$$

If $\nu(s) \neq \pm\frac{\pi}{2}$, then $\cos\nu(s) \neq 0$ and $\mathcal{R}(\gamma)$ holds. So, to prove the first part of the statement, we must show that if ν reaches the value $-\frac{\pi}{2}$, the condition $\mathcal{R}(\gamma)$ also holds. Indeed, if $\nu(s_0) = -\frac{\pi}{2}$ for some $s_0 \in I$, then $b_{s_0}(p) < 0$ and, therefore, we have $\mathcal{R}(\gamma)$.

Suppose now that for some $s_1 \in I$, $\nu(s_1) = \frac{\pi}{2}$. The condition $b_{s_1}(p) \neq 0$ can be written as

$$p \neq \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\nu'(s_1) + \kappa_g^{\alpha}(s_1)}\right).$$

From this follows directly the hypothesis on p of the statement just by using that

$$\sup_{s\in I}\big(\nu'(s) + \kappa_g^{\alpha}(s)\big) \leq \sup_{s\in I}\nu'(s) + \sup_{s\in I}\kappa_g^{\alpha}(s)$$

and

$$\inf_{s\in I}\big(\nu'(s) + \kappa_g^{\alpha}(s)\big) \geq \inf_{s\in I}\nu'(s) + \inf_{s\in I}\kappa_g^{\alpha}(s).$$

□

Remark 5.5. The bounds (5.7) and (5.8) on p of Theorem 5.4 both avoid singularities on the generated curves. Nevertheless, since a small p is usually considered, the bound of interest is (5.7). In fact, if $\nu(s) = \omega$ is constant, that bound is the condition of regularity for parallel curves in constant curvature surfaces:

$$p < \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\kappa_g^{\sup}}\right).$$

Of course, the limit

$$\lim_{K\to 0} \frac{1}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}}{\kappa_g^{\sup}}\right) = \frac{1}{\kappa_{\sup}}$$

gives us the bound for having regular parallel curves in the plane (see [30]), where here $\kappa_{\sup}$ is the supremum curvature of the initial curve. It is also the condition for regularity of pipe surfaces (see [20], p. 399).

Remark 5.6. In the case of Holditch curves, a sufficiently small chord length ℓ will make $\nu(s) \neq \frac{\pi}{2}$ for all $s \in I$. This means that all the p-Holditch curves of a simple regular curve for a sufficiently small chord length are always regular. Otherwise—if the chord length is so big—a sufficiently small choice of p can avoid singularities.

Before the main result—Lemma 5.12—the definition of a frame adapted to the generated curve is needed. The angle between the geodesic chord and the tangent of the generated curve will be relevant and some results on it are also needed.

5.3.2 Generalization of the envelope theorem

Here, a generalization of the envelope theorem is stated and proved using the introduced techniques of non-Euclidean geometry. Visualize in Figure 5.4 the setting of the next result.

Theorem 5.7. *If $\eta(s)$ is the angle between $\gamma'(s)$ and the geodesic λ_s, then*

$$\|\gamma'(s)\| \cos\eta(s) = p'(s) + \cos\nu(s).$$

Proof. On the one hand,

$$\begin{aligned}\gamma'(s) = &-p'(s)\sqrt{K}\sin\big(\sqrt{K}\,p(s)\big)\,\alpha(s) + p'(s)\cos\big(\sqrt{K}\,p(s)\big)\,\beta(s)\\ &+ \cos\big(\sqrt{K}\,p(s)\big)\,\alpha'(s) + \frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}}\,\beta'(s). \qquad (5.9)\end{aligned}$$

On the other hand,

$$\lambda_s'\big(p(s)\big) = -\sqrt{K}\sin\big(\sqrt{K}\,p(s)\big)\,\alpha(s) + \cos\big(\sqrt{K}\,p(s)\big)\,\beta(s).$$

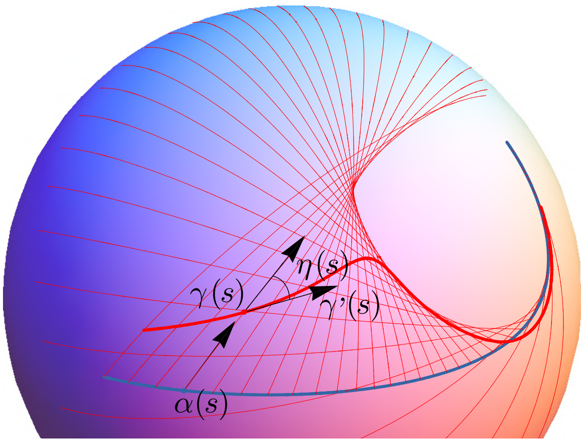

Figure 5.4: Representation of the angle $\eta(s)$ of Theorem 5.7.

Thus, we can compute

$$\mathcal{B}\Big(\gamma'(s), \lambda'_s\big(p(s)\big)\Big) = p'(s) + \cos\nu(s), \tag{5.10}$$

by using that $\mathcal{B}\big(\alpha'(s), \beta(s)\big) = -\mathcal{B}\big(\alpha(s), \beta'(s)\big) = \cos\nu(s)$. Finally, note that $\gamma'(s)$ and $\lambda'_s\big(p(s)\big)$ are space-like vectors, so there exists $\eta(s) \in [0, \pi]$ (the space-like angle between both vectors) such that

$$\mathcal{B}\Big(\gamma'(s), \lambda'_s\big(p(s)\big)\Big) = \big\|\gamma'(s)\big\| \cos\eta(s). \tag{5.11}$$

From (5.10) and (5.11), the result follows. □

Notation 5.8. If t is the arc-length parameter of a curve γ, for any parameter change $t \mapsto s$,

$$\mathrm{d}t = \big\|\gamma'(t)\big\|\,\mathrm{d}s.$$

For any function $F : I \to \mathbb{R}$, adopt the following notation:

$$\int_{\gamma^*} F(s)\,\mathrm{d}t := \int_I F(s)\,\big\|\gamma'(s)\big\|\,\mathrm{d}s.$$

As a corollary of Theorem 5.7, next result is deduced, which was given in [95].

Theorem 5.9. *Let s be the arc-length parameter of α and $L = \mathcal{L}(\alpha)$. If $\beta(s)$ is tangential to $\alpha(s)$ for each $s \in I$, then*

$$\int_{\gamma^*} \cos\eta(s)\,\mathrm{d}t = L + \big(p(L) - p(0)\big),$$

where $\eta(s)$ is the angle between $\gamma'(s)$ and the corresponding tangent geodesic and t is the arc-length parameter of γ.

Proof. Since we have that $\nu(s) = 0$ for all $s \in I$, then by Theorem 5.7,

$$\|\gamma'(s)\| \cos\eta(s) = 1 + p'(s).$$

Integrating in s we get

$$\int_I \cos\eta(s) \, \|\gamma'(s)\| \, \mathrm{d}s = L + \big(p(L) - p(0)\big),$$

which can be rewritten as in the statement with the arc-length parameter of γ. □

Notice that Theorem 5.9 generalizes the envelope curve theorem (see [7], p. 167): in a geodesic line variation that cuts orthogonally a curve γ and with envelope α, the increase of the geodesic distance from α to γ is equal to the corresponding arc of the envelope. Indeed, if $\eta(s) = \frac{\pi}{2}$, then Theorem 5.9 ensures that

$$L = p(0) - p(L).$$

Visualize the envelope theorem in M^K in Figure 5.5.

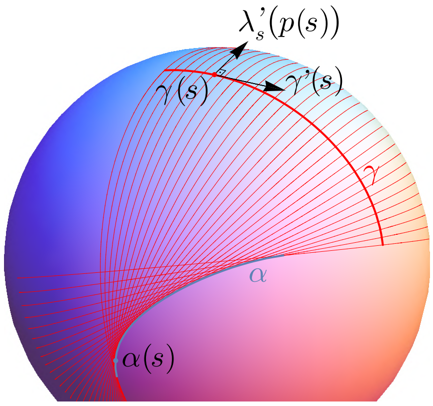

Figure 5.5: Representation of the envelope theorem in M^K.

5.3.3 Orthonormal frame adapted to the generated curve

The natural orthonormal frame adapted to the generated curve (recall the definition of a normal vector in Section 4.2.2) is

$$\mathcal{F}^*(s) = \Big\{ \sqrt{|K|}\,\gamma(s),\, \mathbf{t}_g(s),\, \sqrt{|K|}\,(\gamma \wedge \mathbf{t}_g)(s) \Big\},$$

where $\mathbf{t}_g$ is the tangent of the geodesic λ_s at $\gamma(s)$:

$$\mathbf{t}_g(s) = \lambda_s'\big(p(s)\big) = -\sqrt{K}\,\sin\big(\sqrt{K}\,p(s)\big)\,\alpha(s) + \cos\big(\sqrt{K}\,p(s)\big)\,\beta(s).$$

A straightforward computation shows that

$$\sqrt{|K|}\,(\gamma \wedge \mathbf{t}_g)(s) = -\frac{B_s(u)}{b_s(u)} = -\mathbf{v}(s),$$

where $\mathbf{v}(s)$ is defined in Equation (5.4).

The variation of $\mathcal{F}^*$ is given by the matrix

$$\sqrt{|K|}\begin{pmatrix} 0 & p'(s)+\cos\nu(s) & -b_s\big(p(s)\big) \\ -\operatorname{sgn}(K)\,\big(p'(s)+\cos\nu(s)\big) & 0 & -\frac{b_s'\big(p(s)\big)}{\sqrt{|K|}} \\ \operatorname{sgn}(K)\,b_s\big(p(s)\big) & \frac{b_s'\big(p(s)\big)}{\sqrt{|K|}} & 0 \end{pmatrix}.$$

Suppose now that $\eta(s)\in\mathbb{R}$ is the oriented angle function (according to the orientation of M^K) from $\gamma'(s)$ to $\mathbf{t}_g(s)$ in $T_{\gamma(s)}M^K$ (see Figure 5.6). Thus, by definition,

$$\mathbf{t}_g(s)=\cos\eta(s)\,\frac{\gamma'(s)}{\|\gamma'(s)\|}+\sin\eta(s)\,\sqrt{|K|}\,\gamma\wedge\frac{\gamma'(s)}{\|\gamma'(s)\|}.$$

From this, the tangent vector of γ can be written as

$$\frac{\gamma'(s)}{\|\gamma'(s)\|}=\cos\eta(s)\,\mathbf{t}_g(s)-\sin\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s). \tag{5.12}$$

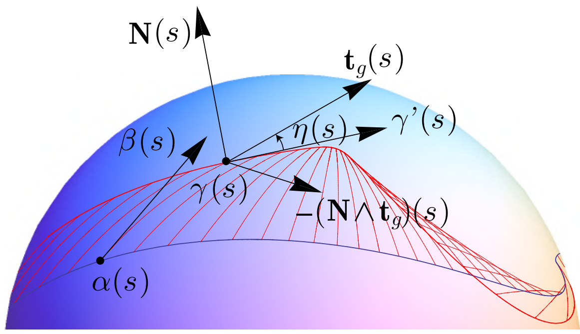

Figure 5.6: Definition of the oriented angle $\eta(s)$ from $\gamma'(s)$ to $\mathbf{t}_g(s)$.

Theorem 5.10. *If $\eta(s)$ is the oriented angle from $\gamma'(s)$ to the geodesic λ_s, then*

$$b_s\big(p(s)\big)=\big\|\gamma'(s)\big\|\sin\eta(s). \tag{5.13}$$

Proof. On the one hand, from Equation (5.12),

$$\mathcal{B}\left(\frac{\gamma'(s)}{\|\gamma'(s)\|},\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\right)=-\sin\eta(s). \tag{5.14}$$

On the other hand, since $\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)=-\mathbf{v}(s)$, it is straightforward to compute

$$\mathcal{B}\Big(\gamma'(s),\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\Big)=-b_s\big(p(s)\big) \tag{5.15}$$

with (5.9) and (5.4). The result follows from (5.14) and (5.15). $\square$

Corollary 5.11. *Suppose that $\mathcal{S}(\alpha)$ and $\mathcal{R}(\gamma)$ holds. Let $\eta(s)$ be the oriented angle from $\gamma'(s)$ to the geodesic λ_s at $\gamma(s)$. Given $s \in I$,*

1. *$p'(s) + \cos\nu(s) = 0$ if and only if $\eta(s) = \pm\frac{\pi}{2}$.*
2. *$b_s\big(p(s)\big) = 0$ if and only if $\eta(s) = 0, \pi$.*

Proof. Since $\big\|\gamma(s)\big\| \neq 0$ for all $s \in I$, the result follows immediately from Theorems 5.7 and 5.10. □

5.3.4 Main results on generated curves

In this section, the main results on generated curves will be derived. Some parts of this section can be found in [64].

Follow the notation below for the total geodesic curvature of a curve $\phi : I \to M^K$ with trace ϕ^*:

$$\int_{\phi^*} \kappa_g^\phi := \int_I \kappa_g^\phi(t)\,\big\|\phi'(t)\big\|\,\mathrm{d}t.$$

Recall that the total geodesic curvature is invariant by reparameterizations (up to a sign if the orientation of the curve or of the manifold is changed).

Now, the relation between the geodesic curvatures of both curves, α and the generated γ, can be stated. It is written in terms of the angles ν (see Section 5.2) and η (see Section 5.3.3).

Lemma 5.12. *If $\mathcal{S}(\alpha)$ holds and the generated curve γ is regular—hypothesis $\mathcal{R}(\gamma)$—then*

$$\int_{\gamma^*} \kappa_g^\gamma = \sqrt{K}\int_I \sin\big(\sqrt{K}\,p(s)\big)\sin\nu(s)\,\mathrm{d}s + \int_I \cos\big(\sqrt{K}\,p(s)\big)\,\big(\nu'(s) + \kappa_g^\alpha(s)\big)\,\mathrm{d}s - \int_I \eta'(s)\,\mathrm{d}s,$$

where κ_g^γ and κ_g^α are the geodesic curvatures of γ and α, respectively.

Proof. Mainly, it is a matter of computation of the geodesic curvature of γ with the expression (4.8). Since we are going to integrate, it is sufficient to compute

$$\sqrt{|K|}\,\frac{\mathcal{B}(\gamma, \gamma' \wedge \gamma'')(s)}{\big\|\gamma'(s)\big\|^2}.$$

(Notice that γ may not be arc-length parameterized.) With the frame $\mathcal{F}$, the computation of the geodesic curvature leads to a wide expression. Hence, it is convenient to make the same computations but using an orthonormal frame adapted to γ, such as $\mathcal{F}^*$ (see Section 5.3.3). As in Equation (5.12), we can write

$$\frac{\gamma'(s)}{\|\gamma'(s)\|} = \cos\eta(s)\,\mathbf{t}_g(s) - \sin\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s). \tag{5.16}$$

Differentiating (5.16), we get

$$\frac{\gamma''(s)}{\|\gamma'(s)\|} + \left(\frac{1}{\|\gamma'(s)\|}\right)'\gamma'(s) \tag{5.17}$$

$$\begin{aligned}
&= \eta'(s)\,\big(-\sin\eta(s)\,\mathbf{t}_g(s) - \cos\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big)\\
&\quad + \cos\eta(s)\,\mathbf{t}_g'(s) - \sin\eta(s)\,\big(\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big)'\\
&= \eta'(s)\,\big(-\sin\eta(s)\,\mathbf{t}_g(s) - \cos\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big)\\
&\quad + b_s'\big(p(s)\big)\,\big(\sin\eta(s)\,\mathbf{t}_g(s) + \cos\eta(s)\,\sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\big)\\
&\quad + \big(\text{term in } \sqrt{|K|}\,\gamma(s)\big).
\end{aligned} \tag{5.18}$$

Hence, making the cross product of (5.16) and (5.17),

$$\begin{aligned}
\frac{(\gamma'\wedge\gamma'')(s)}{\|\gamma'(s)\|^2} &= \operatorname{sgn}(K)\,\Big(b_s'\big(p(s)\big) - \eta'(s)\Big)\,\sqrt{|K|}\,\gamma(s)\\
&\qquad + \Big(\text{terms in } \mathbf{t}_g(s) \text{ and } \sqrt{|K|}\,(\gamma\wedge\mathbf{t}_g)(s)\Big).
\end{aligned}$$

Therefore,

$$\sqrt{|K|}\,\frac{\mathcal{B}\big(\gamma,\gamma'\wedge\gamma''\big)(s)}{\|\gamma'(s)\|^2} = b_s'\big(p(s)\big) - \eta'(s). \tag{5.19}$$

Since

$$b_s'\big(p(s)\big) = \sqrt{K}\sin\big(\sqrt{K}\,p(s)\big)\sin\nu(s) + \cos\big(\sqrt{K}\,p(s)\big)\,\big(\nu'(s) + \kappa_g^{\alpha}(s)\big),$$

the result is obtained by integrating (5.19). □

The next step is to translate the expression of Lemma 5.12 in terms of areas when α and γ are both simple and closed and the length $p(s)$ is constant (Lemma 5.16). For that, Gauss–Bonnet theorem will be used (Remark 5.15).

Hypothesis $\mathcal{CR}(\alpha,\gamma)$. The hypothesis $\mathcal{S}(\alpha)$ holds and in addition α is closed and the functions $\beta(s)$ and $p(s)$ can be extended continuously by periodicity (so γ is also closed). Moreover, assume the generated curve γ to be positively oriented, simple and regular—thus, $\mathcal{R}(\gamma)$ holds.

Since the regularity and simpleness of the generated curves is dependent on the chosen length of the moving chord, the next definition makes sense (recall that it has been already given thinking of generated planar curves: Definition 3.51).

Definition 5.13 (Admissible length)**.** A constant length p is said to be *admissible* if the generated curve γ is regular and simple.

The global version of Gauss–Bonnet theorem for 2-dimensional Riemannian manifolds is the next (see [35]).

Theorem 5.14 (Gauss–Bonnet theorem)**.** *Let M be an orientable Riemannian manifold of dimension 2. Let R be a compact region of M with boundary ∂R composed by the union of a finite number of positively oriented piecewise-regular simple closed curves $c_1, c_2, \ldots, c_n$ that do not intersect each other. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ be the external angles of c_i. If K is the Gauss curvature of M and κ_g is the geodesic curvature of ∂R, then*

$$\int_R K \,\mathrm{d}A + \int_{\partial R} \kappa_g + \sum_{i=1}^{p} \epsilon_i = 2\pi\, \chi(R),$$

where $\mathrm{d}A$ is the area element of R and $\chi(R)$ is the Euler–Poincaré characteristic of R.

Remark 5.15. Given a compact region R of M^K with boundary defined by the trace of a positively oriented regular simple closed curve $\phi : I \to \partial R$, Gauss–Bonnet theorem (Theorem 5.14) implies

$$\int_{\phi^*} \kappa_g^{\phi} = 2\,\pi - K\mathcal{A}(\phi). \tag{5.20}$$

Lemma 5.16. *If $\mathcal{CR}(\alpha,\gamma)$ holds and γ is the generated curve for a chord of constant length $p(s) = p$, then*

$$\begin{aligned}\mathcal{A}(\gamma) = &-\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}} \int_I \sin\nu(s)\,\mathrm{d}s \\ &+ \frac{2\,\pi\,n}{K}\Big(1-\cos\big(\sqrt{K}\,p\big)\Big) + \cos\big(\sqrt{K}\,p\big)\mathcal{A}(\alpha),\end{aligned} \tag{5.21}$$

where n is the number of revolutions done by the geodesic chord in M^K.

Proof. Since $p(s) = p$, the expression of Lemma 5.12 takes the form

$$\begin{aligned}\int_{\gamma^*} \kappa_g^{\gamma} = &\sqrt{K}\,\sin\big(\sqrt{K}\,p\big)\int_I \sin\nu(s)\,\mathrm{d}s \\ &+ \cos\big(\sqrt{K}\,p\big)\int_I \big(\nu'(s)+\kappa_g^{\alpha}(s)\big)\,\mathrm{d}s - \int_I \eta'(s)\,\mathrm{d}s.\end{aligned}$$

Since α and γ are positively oriented, we have

$$\int_I \nu'(s)\,\mathrm{d}s = \int_I \eta'(s)\,\mathrm{d}s = 2\,\pi\,m,$$

where m is the number of revolutions of the moving chord with respect to the tangent vector of α (or the same for γ). Thus,

$$\int_{\gamma^*} \kappa_g^\gamma = \sqrt{K}\,\sin\big(\sqrt{K}\,p\big)\int_I \sin\nu(s)\,\mathrm{d}s + 2\,\pi\,m\,\Big(\cos\big(\sqrt{K}\,p\big) - 1\Big) + \cos\big(\sqrt{K}\,p\big)\int_{\alpha^*}\kappa_g^\alpha.$$

Using Equation (5.20) to both α and γ and arranging the expression we get:

$$\mathcal{A}(\gamma) = -\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}}\int_I \sin\nu(s)\,\mathrm{d}s + \frac{2\,\pi\,(m+1)}{K}\Big(1 - \cos\big(\sqrt{K}\,p\big)\Big) + \cos\big(\sqrt{K}\,p\big)\mathcal{A}(\alpha).$$

Finally, notice that the tangent vector of α (or γ) with respect to a parallel tangent vector field along such a curve makes just a positive complete revolution because both α and γ are simple and positively oriented curves. Therefore, the number n of chord revolutions is equal to $m+1$. □

Example 5.17. Consider in $\mathbb{S}^2(K)$ for $K > 0$ the motion of the geodesic chord shown in Figure 5.7, where one of its endpoints goes along a parallel α. On the one hand, it turns out that such a chord makes a negative revolution with respect to the tangent vector of α. On the other hand, the tangent vector of α makes a positive full revolution. Thus, the total number of chord revolutions is $n = 0$.

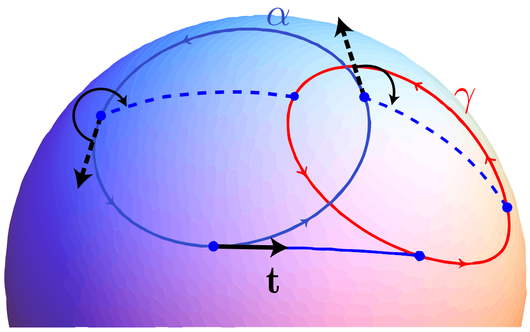

Figure 5.7: The motion of a geodesic chord with its endpoints into two closed regular simple curves α and γ. The oriented angle function from the tangent $\mathbf{t}$ of α to the geodesic chord makes a negative full turn ($m = -1$).

See in Figure 5.8 another example of a geodesic chord motion along the same parallel α. In this case, the geodesic chord makes $m = -3$ complete revolutions with respect to the tangent vector of α. Nevertheless, since its tangent vector makes a full positive revolution, the chord revolutions are $n = m + 1 = -2$.

Notice that in both examples the curves α and γ are positively oriented, so that Lemma 5.16 can be used to relate their areas with the motion of the geodesic chord.

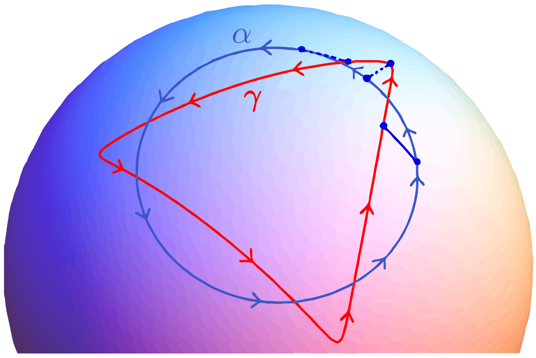

Figure 5.8: Another motion of a geodesic chord with its endpoints into two closed regular simple curves α and γ. In this case, the chord makes two complete negative rotations with respect to a parallel tangent vector field along α.

Now, from Equation (5.21) of Lemma 5.16 some results can be deduced by taking particular situations for the generated curve γ, assuming that such a curve is regular and simple (admissible chord length).

Parallel curves

If $p(s) = p$ and $\nu(s) = \omega$ are both constant, the generated curve γ is a parallel curve to α at a distance p in a direction maintaining an angle ω with α (see Definition 4.39).

The next result is the generalization to constant curvature surfaces of Steiner's formula for the area of parallel curves according to a constant angle (see [93] and [94]).

Theorem 5.18 (Steiner's formula for the area of a parallel curve in M^K)**.** *Let α be a positively oriented regular simple closed curve of finite length in a 2-dimensional manifold of constant curvature K. Let γ be the parallel curve to α at a constant distance p maintaining a constant angle ω with α. If p is admissible, then*

$$\begin{aligned}\mathcal{A}(\gamma) = -\mathcal{L}(\alpha)\,&\frac{\sin\omega\,\sin\left(\sqrt{K}\,p\right)}{\sqrt{K}} \\ &+ \frac{2\pi}{K}\left(1-\cos\left(\sqrt{K}\,p\right)\right) + \cos\left(\sqrt{K}\,p\right)\mathcal{A}(\alpha). \qquad (5.22)\end{aligned}$$

Proof. The result follows just by taking the angle function $\nu(s) = \omega$ constant in Lemma 5.16. Notice that $n = 1$ in this case. $\square$

Remark 5.19. Making $K \to 0$ in Equation (5.22), Steiner's formula in the plane for the area of parallel curves according to a constant angle (see Theorem 3.52) is obtained:

$$\mathcal{A}(\gamma) = \mathcal{A}(\alpha) - p\,\mathcal{L}(\alpha)\,\sin\omega + \pi\,p^2.$$

There is also the extension of Steiner's formula for the length of a parallel curve to constant curvature surfaces (see [93] and [94]).

Theorem 5.20 (Steiner's formula for the length of a parallel curve in M^K)**.** *Let α be a positively oriented regular simple closed curve of finite length in a 2-dimensional manifold of constant curvature K. Let γ be the parallel curve to α found orthogonally at a constant distance p. If p is admissible and sufficiently small, then*

$$\mathcal{L}(\gamma) = \mathcal{L}(\alpha)\, \cos\big(\sqrt{K}\, p\big) \pm \frac{\sin\big(\sqrt{K}\, p\big)}{\sqrt{K}} \big(2\pi - K\, \mathcal{A}(\alpha)\big). \tag{5.23}$$

The sign $\pm$ is $-$ for inner offsets and $+$ for outer offsets.

Proof. The length of the generated curve γ can be computed as

$$\mathcal{L}(\gamma) = \int_I \big\|\gamma'(s)\big\|\, \mathrm{d}s.$$

We have that $p(s) = p$ and $\nu(s) = \omega = \pm\frac{\pi}{2}$ are both constant, so the first term of the right hand side of Equation (5.6) disappears and we have a perfect square. Since p is admissible, hypothesis $\mathcal{R}(\gamma)$ holds.

On the one hand, suppose $p < \frac{\pi}{2\sqrt{K}}$ if $K > 0$. On the other hand, consider p small enough to have constant sign in $b_s(p)$—positive for inner parallels (for which $\omega = \pi/2$ and notice that $b_s(0) = 1$) and negative for outer parallels (for which $\omega = -\pi/2$ and $b_s(0) = -1$). In that case,

$$\mathcal{L}(\gamma) = \int_0^{\mathcal{L}(\alpha)} \big|b_s(p)\big|\, \mathrm{d}s = \int_0^{\mathcal{L}(\alpha)} \left(\cos\big(\sqrt{K}\, p\big) \pm \kappa_g(s)\, \frac{\sin\big(\sqrt{K}\, p\big)}{\sqrt{K}} \right) \mathrm{d}s.$$

Using (5.20), the result is obtained. □

Remark 5.21. Of course, the classical planar Steiner formula for the length of offset curves (see Theorem 1.8),

$$\mathcal{L}(\gamma) = \mathcal{L}(\alpha) \pm 2\,\pi\, p,$$

follows by making $K \to 0$ in Equation (5.23).

Constant width curves

Suppose that the angle function $\nu(s)$ is a constant right angle and that the other endpoint of the chord always lies in α at the constant distance $p(s) = \ell$. That setting is the definition of α being a curve of constant width ℓ (see Definition 4.44).

Now, Barbier's theorem for constant width curves in constant curvature surfaces (see [91]) can be deduced.

Theorem 5.22 (Barbier's theorem in M^K)**.** *Let α be a simple and regular curve of constant width ℓ in a 2-dimensional manifold of constant curvature K. Then*

$$\mathcal{L}(\alpha) = \frac{\sqrt{K}}{\sin\bigl(\sqrt{K}\,\ell\bigr)}\left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right)\Bigl(1-\cos\bigl(\sqrt{K}\,\ell\bigr)\Bigr). \tag{5.24}$$

Proof. Suppose α to be positively oriented. The result follows just by taking $\nu(s) = \pi/2$ and $p(s) = \ell$ in Lemma 5.16 by noticing that, therefore, $\mathcal{A}(\alpha) = \mathcal{A}(\gamma)$ and $n = 1$. □

Remark 5.23. Making $K \to 0$ in Equation (5.24), Barbier's theorem for planar curves of constant width ℓ (see Theorem 1.19) is found:

$$\mathcal{L}(\alpha) = \pi\,\ell.$$

Remark 5.24. The expression (5.24) of Barbier's theorem in M^K is sometimes written as follows:

$$\mathcal{L}(\alpha) = \frac{2\pi - K\,\mathcal{A}(\alpha)}{\sqrt{K}\sin\omega}\tan\left(\sqrt{K}\,\frac{d}{2}\right). \tag{5.25}$$

Holditch's extensions

In this section, the generated curve γ will be a Holditch curve. Notice that the hypothesis $\mathcal{S}(\alpha)$ demands, for each $\alpha(s)$, to be only one direction $\beta(s) \in T_{\alpha(s)}M^K$ to construct the generated curve γ. Implicitly this hypothesis avoids retrograde motion in the moving chord in a Holditch setting for an initial curve.

Restrictions on the length of the chord in order to avoid retrograde movements in the plane were commented in Section 3.1.4. Analogous results on constant curvature surfaces are hoped, but a more detailed work is needed. From now on in this section, assume that no retrograde movements appear for the chosen chord length as a hypothesis.

Next, Lemma 5.16 is used to give a new proof of Holditch's theorem in 2-dimensional constant curvature manifolds for two initial curves α and $\bar{\alpha}$. See [83] for an equivalent expression of (5.26) as a determinant. In Figure 5.9 the setting is represented.

Theorem 5.25 (Holditch's theorem for two initial curves in M^K)**.** *Let α and $\bar{\alpha}$ be two positively oriented closed simple and regular curves in a 2-dimensional manifold of constant curvature K. Suppose that a chord of constant length ℓ moves forwards with each of its endpoints in each curve such that n chord revolutions (according to the orientation of M^K) are done. A point dividing the chord into two parts of lengths p and q will describe another closed curve γ. If p is admissible, then*

$$\mathcal{A}(\gamma) = \frac{\sin\bigl(\sqrt{K}\,q\bigr)\mathcal{A}(\alpha) + \sin\bigl(\sqrt{K}\,p\bigr)\mathcal{A}(\bar{\alpha})}{\sin\bigl(\sqrt{K}\,(p+q)\bigr)} - n\,C_{p,q}(K), \tag{5.26}$$

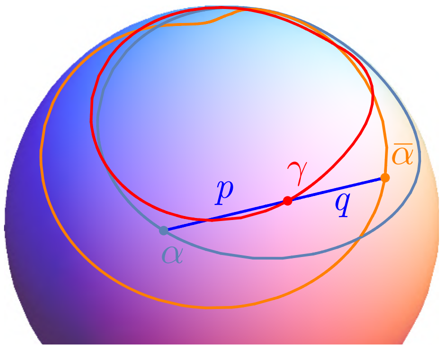

Figure 5.9: Two initial closed curves α and $\bar{\alpha}$ and the associated Holditch curve for a chord of constant length $\ell = p + q$ moving a full revolution without retrograde motion.

where n is the number of chord revolutions and

$$C_{p,q}(K) := 4\pi \, \frac{\sin\left(\sqrt{K}\,\frac{p}{2}\right)\sin\left(\sqrt{K}\,\frac{q}{2}\right)}{K\cos\left(\sqrt{K}\,\frac{p+q}{2}\right)}.$$

Proof. Let's use Lemma 5.16. Suppose now that at a constant distance p in the chord from α a curve γ is generated but also at a constant distance $\ell \geq p$ the curve $\bar{\alpha}$ is described. So, we have

$$\begin{aligned}\mathcal{A}(\gamma) = &-\frac{\sin\left(\sqrt{K}\,p\right)}{\sqrt{K}} \int_I \sin\nu(s)\;\mathrm{d}s + \frac{2\,\pi\,n}{K}\left(1-\cos\left(\sqrt{K}\,p\right)\right)\\ &+\cos\left(\sqrt{K}\,p\right)\mathcal{A}(\alpha)\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\mathcal{A}(\bar{\alpha}) = &-\frac{\sin\left(\sqrt{K}\,\ell\right)}{\sqrt{K}} \int_I \sin\nu(s)\;\mathrm{d}s + \frac{2\,\pi\,n}{K}\left(1-\cos\left(\sqrt{K}\,\ell\right)\right)\\ &+\cos\left(\sqrt{K}\,\ell\right)\mathcal{A}(\alpha).\end{aligned} \tag{5.28}$$

If we multiply Equation (5.27) by $\sin\left(\sqrt{K}\,\ell\right)$ and subtract Equation (5.28) multiplied by $\sin\left(\sqrt{K}\,p\right)$, we get

$$\begin{aligned}\sin\left(\sqrt{K}\,\ell\right)\mathcal{A}(\gamma) - \sin\left(\sqrt{K}\,p\right)\mathcal{A}(\bar{\alpha}) &= \mathcal{A}(\alpha)\,\sin\left(\sqrt{K}\,(\ell-p)\right)\\ &+\frac{2\,\pi\,n}{K}\left(\sin\left(\sqrt{K}\,\ell\right)-\sin\left(\sqrt{K}\,p\right)-\sin\left(\sqrt{K}\,(\ell-p)\right)\right).\end{aligned}$$

Arranging and noticing that

$$\frac{\sin\left(\sqrt{K}\,p\right)-\sin\left(\sqrt{K}\,\ell\right)+\sin\left(\sqrt{K}\,(\ell-p)\right)}{\sin\left(\sqrt{K}\,\ell\right)} = 2\,\frac{\sin\left(\sqrt{K}\,\frac{p}{2}\right)\sin\left(\sqrt{K}\,\frac{\ell-p}{2}\right)}{\cos\left(\sqrt{K}\,\frac{\ell}{2}\right)},$$

the expression of the statement is obtained. ☐

Remark 5.26. Making $K \to 0$ in (5.26), Woolhouse's extension of Holditch's theorem (see Theorems 2.2 and 3.55) for two initial curves in the plane is obtained:

$$\mathcal{A}(\gamma) = \frac{q\,\mathcal{A}(\alpha) + p\,\mathcal{A}(\bar{\alpha})}{p+q} - n\,\pi\,p\,q.$$

Notice that n agrees with the number of planar chord revolutions.

The constant $C_{p,q}(K)$ is called the *Holditch constant* because it generalizes the Holditch constant of the planar case:

$$\lim_{K\to 0} C_{p,q}(K) = \pi\,p\,q.$$

The interpretation of the constant $C_{p,q}(K)$ will be addressed in Section 5.4.3.

As a corollary of Theorem 5.25, the statement of Holditch's theorem in constant curvature manifolds can be deduced (see [91], [95] or [83]). In Figure 5.10 this setting is represented.

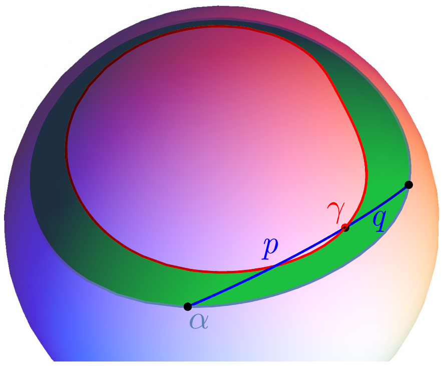

Figure 5.10: A closed curve, α, and its Holditch curve, γ, generated by a chord of constant length $p+q$. The Holditch area (shaded) is the difference between the areas of both closed curves.

Theorem 5.27 (Holditch's theorem in M^K)**.** *Let α be a positively oriented closed simple and regular curve in a 2-dimensional manifold of constant curvature K. Suppose that a chord of constant length ℓ moves forwards with both endpoints lying in α such that a full chord revolution (according to the orientation of M^K) is done. A point dividing the chord into two parts of lengths p and q will describe another closed curve γ. If p is admissible, then*

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = C_{p,q}(K) \cdot \left(1 - \frac{K\mathcal{A}(\alpha)}{2\pi}\right), \tag{5.29}$$

where

$$C_{p,q}(K) := 4\pi\,\frac{\sin\!\left(\sqrt{K}\,\frac{p}{2}\right)\sin\!\left(\sqrt{K}\,\frac{q}{2}\right)}{K\cos\!\left(\sqrt{K}\,\frac{p+q}{2}\right)}.$$

Proof. We will use Theorem 5.25 when $\bar{\alpha}$ is another parameterization of the curve α. Since α is simple and both endpoints of the moving chord must be on α, we have $n = 1$. Thus, $\mathcal{A}(\alpha) = \mathcal{A}(\bar{\alpha})$ and the formula (5.26) become

$$\mathcal{A}(\gamma) = \frac{\sin\big(\sqrt{K}\, q\big) + \sin\big(\sqrt{K}\, p\big)}{\sin\big(\sqrt{K}\,(p+q)\big)}\, \mathcal{A}(\alpha) - C_{p,q}(K).$$

Therefore, we get

$$\mathcal{A}(\gamma) - \mathcal{A}(\alpha) = C_{p,q}(K) \cdot \frac{K}{2\pi}\, \mathcal{A}(\alpha) - C_{p,q}(K),$$

from which the expression of the statement is deduced. $\square$

Remark 5.28. Of course, making $K \to 0$ in the formula (5.29), the classical Holditch theorem in the plane is obtained:

$$\mathcal{A}(\alpha) - \mathcal{A}\big(\gamma\big) = \pi\, p\, q.$$

Notice that in the general statement the Holditch area also depends on the area of the initial curve besides the curvature K and the chosen lengths p and q in the chord.

Bicycle curves

If α is the rear wheel of a bicycle (as defined in Section 4.3.3), then the generated curve at a constant length ℓ for tangent geodesic directions, γ, is the front wheel (see Figure 5.11). From Lemma 5.16, it is easy to get an expression for the swept out area in a closed tire-track movement.

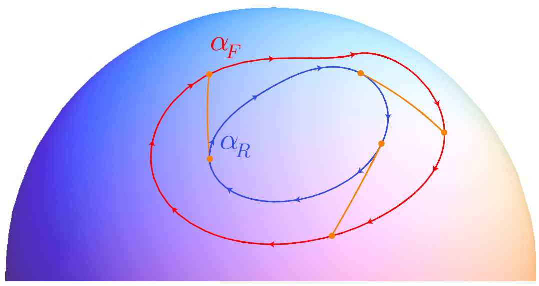

Figure 5.11: The tire track of the rear wheel, α_R, and the front one, α_F. The front wheel is the generated curve from α_R at a distance the wheelbase of the bicycle.

Theorem 5.29. *If α_R and α_F are two regular simple closed planar curves describing the rear and the front wheel, respectively, of a bicycle of constant length ℓ in M^K, then*

$$\mathcal{A}(\alpha_F) - \mathcal{A}(\alpha_R) = 2\pi\, \frac{1 - \cos\big(\sqrt{K}\,\ell\big)}{K}.$$

Proof. It is just a matter of applying Lemma 5.16 taking $\nu(s) = 0$ for all $s \in I$ (so $n = 1$). $\square$

Of course, since

$$\lim_{K \to 0} \frac{1 - \cos\left(\sqrt{K}\,\ell\right)}{K} = \frac{\ell^2}{2},$$

the formula of Theorem 5.29 generalizes Theorem 3.58.

5.4 On Holditch's theorem in M^K

5.4.1 Definition of Holditch curve in M^K

Analogously as done in the plane in Section 3.1, the notion of Holditch function can also be defined for curves in M^K (but using the corresponding metric of M^K). The reader is referred to this section for such a definition. Moreover, the definition of *retrograde motion* can be done according to these functions as in Definition 3.2. In the same way, a parametric definition of the Holditch curve is possible and will be given below.

Proposition 5.30. *Given a geodesic chord in M^K of length ℓ with endpoints A and B, a point C placed in such a chord at a distance p from A can be written as*

$$C = \frac{\sin\big(\sqrt{K}\,(\ell - p)\big)\, A + \sin\big(\sqrt{K}\,p\big)\, B}{\sin\big(\sqrt{K}\,\ell\big)}.$$

Proof. By Proposition 4.36, the unit speed geodesic γ that passes through A and B such that $\gamma(0) = A$ has parameterization

$$\gamma(s) = \cos\big(\sqrt{K}\,s\big)\, A + \frac{\sin\big(\sqrt{K}\,s\big)}{\sqrt{K}}\,\mathbf{v},$$

where

$$\mathbf{v} = -\frac{A \wedge (A \wedge B)}{\big\|A \wedge (A \wedge B)\big\|} = \frac{\sqrt{K}}{\sin\big(\sqrt{K}\,\ell\big)}\Big(B - \cos\big(\sqrt{K}\,\ell\big)\, A\Big).$$

Thus, we have

$$\gamma(s) = \frac{\sin\big(\sqrt{K}\,(\ell - s)\big)\, A + \sin\big(\sqrt{K}\,s\big)\, B}{\sin\big(\sqrt{K}\,\ell\big)}.$$

The equality $C = \gamma(p)$ yields the expression in the statement. □

From the previous result, the definition of Holditch curve in M^K by its parameterization follows immediately.

Definition 5.31 (Holditch curve in M^K)**.** Let $\alpha : I \to M^K$ be a positively oriented curve. Given $\ell > 0$ and $0 \le p \le \ell$, the *Holditch curve of α generated by a geodesic chord of length ℓ at a distance p from the rear end* is defined by

$$H_\alpha(s) = \frac{\sin\big(\sqrt{K}\,(\ell - p)\big)\, \alpha\big(g(s)\big) + \sin\big(\sqrt{K}\,p\big)\, \alpha\big(h(s)\big)}{\sin\big(\sqrt{K}\,\ell\big)}, \tag{5.30}$$

where g is the rear Holditch function and h the front Holditch function, i.e. they are continuous maps such that $\alpha\big(g(s)\big)$ and $\alpha\big(h(s)\big)$ describe the endpoints of the moving geodesic chord.

Indeed, the expression (5.30) generalizes the planar one which allows retrograde motion, namely Equation (3.4), by making $K \to 0$:

$$H_\alpha(s) = \frac{1}{\ell}\Big((\ell - \mathfrak{p})\,\alpha\big(g(s)\big) + \mathfrak{p}\,\alpha\big(h(s)\big)\Big).$$

As in the plane, if $p = \mathfrak{p}/\ell \in [0,1]$, then (5.30):

$$H_\alpha(s) = \frac{\sin\big(\sqrt{K}\,\ell\,(1-p)\big)\,\alpha\big(g(s)\big) + \sin\big(\sqrt{K}\,\ell\,p\big)\,\alpha\big(h(s)\big)}{\sin\big(\sqrt{K}\,\ell\big)},$$

is called the *p-Holditch curve of α for a chord length ℓ.*

5.4.2 Sweeping out an area in M^K

The first extension of Holditch's theorem in the direction of Theorem 5.27 was done by Elliott in [27], but only for the sphere where it was presented together with another theorems of spherical kinematics (see Section 2.5). The first general setting for any constant curvature surface was given originally by Vidal Abascal in [91] and revisited by the same author and Rodeja in [95]. Later on, Santaló in [83] gave a different proof of the same theorem and extended it to n dimensions to continue the approach of Kurita, who in [49] gave the result for volumes in the Euclidean space of n dimensions.

The proof of Vidal Abascal comes from the theory of surfaces. In Santaló's own words: "Vidal Abascal and Rodeja use the methods from the differential geometry of constant curvature surfaces. Although the geometry on these surfaces is equivalent to the non-Euclidean geometry on the plane, we think that it could be of some interest to obtain the same result using the computation methods of this kind of geometry". Thus, in Santaló's paper, a brief introduction to non-Euclidean geometry is given (see Chapter 4) and methods involving differential forms are used.

Note that the classical Holditch formula in the plane, namely $\pi\,p\,q$, is on a difference between areas and it also holds for non-convex curves. Therefore, its natural generalization to constant curvature surfaces should also be on a difference of areas as in Theorem 5.27. Nevertheless, Vidal Abascal in his paper computed the Holditch area as a *swept out area* by the moving chord, which has some disadvantages if the chosen p distance from one end is too long because overlapping area problems have to be managed. Moreover, the fact of finding at the end a theorem on a difference of areas is a consequence of some assumptions on convexity (see Corollary 5.34).

In this section, following the approach of Vidal Abascal, computing areas directly with the area element, another proof of Holditch's theorem in a 2-dimensional constant curvature manifold can be given by taking the advantages of the construction of M^K given in Chapter 4. Nevertheless, as commented, an additional hypothesis to avoid overlapping regions will be taken. This hypothesis can be stated thinking of the normal Jacobi field $B_s(u)$ along the initial curve, defined in Section 5.2, as given below.

Given $s \in I$, if $b_s(0) = \sin\nu(s)$ is positive, then there exists a neighborhood U of 0 where $b_s(u) > 0$ for all $u \in U$. This condition will avoid points in which the normal Jacobi field $B_s(u)$ is zero. Let's write it.

Hypothesis $\mathcal{SR}(\alpha,\gamma)$. The hypothesis $\mathcal{S}(\alpha)$ holds and, in addition, we have $b_s(u) > 0$ for any $s \in I$ and $u \in [0, p(s)]$, $p(s)$ being the length function for the generated curve γ.

Notice that $\mathcal{SR}(\alpha,\gamma)$ implies $\mathcal{R}(\gamma)$. First, general theorems on generated curves will be stated and, afterwards, the Holditch case will be addressed. The hypothesis $\mathcal{SR}(\alpha,\gamma)$ ensures that the movement of the chord is done such that its swept-out area has no overlapping regions (see Figure 5.12).

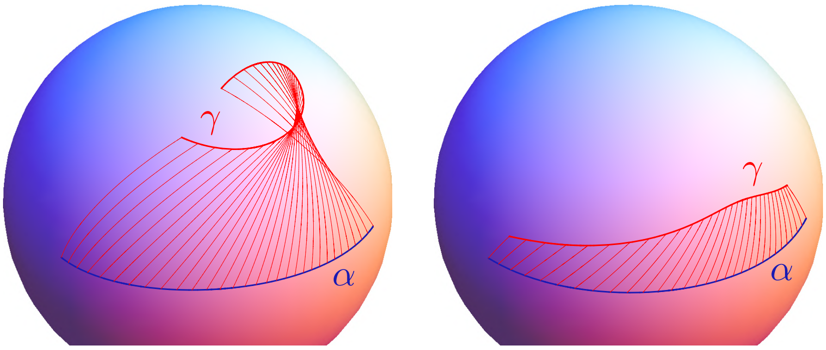

Figure 5.12: On the left, the generated curve γ with a choice of $p(s)$ such that overlapping swept-area problems appear in the chord. On the right, the same but with a choice of $p(s)$ with no overlapping problems.

Theorem 5.32. *Suppose $\mathcal{SR}(\alpha,\gamma)$. The area swept out by the moving chord which produces the generated curve γ from α, for a length function $p(s)$ and directions forming an oriented angle $\nu(s)$ from the tangent vector of α at $\alpha(s)$ to them, is equal to*

$$\int_I \frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}} \sin\nu(s)\,\mathrm{d}s - \frac{1}{K}\int_I \Big(1-\cos\big(\sqrt{K}\,p(s)\big)\Big)\,\big(\nu'(s)+\kappa_g(s)\big)\,\mathrm{d}s.$$

Proof. If we denote

$$V := \big\{(s,u)\,:\, s \in I,\ u \in\,]0, p(s)[\big\},$$

the manifold M^K can locally be seen with a chart $\mathbf{x} : V \to \mathbb{R}^3$ defined by

$$\mathbf{x}(s,u) = \lambda_s(u) = \cos\big(\sqrt{K}\,u\big)\,\alpha(s) + \frac{\sin\big(\sqrt{K}\,u\big)}{K}\,\beta(s),$$

with β being, as in (5.2), the unit direction of the moving chord. Note that $\mathbf{x}(s,0) = \alpha(s)$ and $\mathbf{x}\big(s,p(s)\big) = \gamma(s)$, with γ being the generated curve from α in the direction of β for a length function p.

With this local chart, the area swept out by the geodesic moving chord can be computed as

$$\int_I \int_0^{p(s)} \sqrt{g(s,u)}\, \mathrm{d}u\, \mathrm{d}s,$$

where $g(s,u) = \big(g_{11}\, g_{22} - g_{12}^2\big)(s,u)$.

The coefficients of $\mathbf{x}(s,u)$ on the frame $\mathcal{F}$ are

$$\left(\frac{\cos\big(\sqrt{K}\,u\big)}{\sqrt{|K|}},\ \frac{\sin\big(\sqrt{K}\,u\big)\cos\nu(s)}{\sqrt{K}},\ \frac{\sin\big(\sqrt{K}\,u\big)\sin\nu(s)}{\sqrt{K}}\right).$$

It is easy to compute the coefficients of the derivatives $\mathbf{x}_s(s,u)$ and $\mathbf{x}_u(s,u)$ on $\mathcal{F}$, which are, respectively,

$$\Bigg(-\sqrt{\operatorname{sgn}(K)}\,\sin\big(\sqrt{K}\,u\big)\cos\nu(s),$$
$$\cos\big(\sqrt{K}\,u\big) - \frac{\sin\big(\sqrt{K}\,u\big)\,\big(\nu'(s)+\kappa_g^\alpha(s)\big)\sin\nu(s)}{\sqrt{K}},$$
$$\frac{\sin\big(\sqrt{K}\,u\big)\,\big(\nu'(s)+\kappa_g^\alpha(s)\big)\cos\nu(s)}{\sqrt{K}}\Bigg)$$

and

$$\Big(-\sqrt{\operatorname{sgn}(K)}\sin\big(\sqrt{K}\,u\big),\ \cos\big(\sqrt{K}\,u\big)\cos\nu(s),\ \cos\big(\sqrt{K}\,u\big)\sin\nu(s)\Big).$$

A straightforward calculation yields $g(s,u) = \big(b_s(u)\big)^2$. Thus, by the hypothesis $\mathcal{SR}(\alpha,\gamma)$,

$$\sqrt{g(s,u)} = b_s(u) = \cos\big(\sqrt{K}\,u\big)\sin\nu(s) - \frac{\sin\big(\sqrt{K}\,u\big)}{\sqrt{K}}\,\big(\nu'(s)+\kappa_g^\alpha(s)\big).$$

Hence,

$$\int_0^{p(s)} \sqrt{g(s,u)}\,\mathrm{d}u = \frac{\sin\big(\sqrt{K}\,p(s)\big)}{\sqrt{K}}\sin\nu(s) - \frac{1-\cos\big(\sqrt{K}\,p(s)\big)}{K}\,\big(\nu'(s)+\kappa_g(s)\big).$$

The area of interest is obtained by integrating this expression on I. □

Corollary 5.33. *Suppose that $\mathcal{SR}(\alpha,\gamma)$ holds and, in addition, α is closed, $p(s) = p$ is constant and the function $\beta(s)$ can be extended continuously by periodicity (so γ is also closed). If p is admissible, then the area swept out by the moving chord is equal to*

$$\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}}\int_I \sin\nu(s)\,\mathrm{d}s - \Big(1-\cos\big(\sqrt{K}\,p\big)\Big)\left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right).$$

Proof. Since $p(s) = p$ is constant and by hypothesis $\mathcal{SR}(\alpha, \gamma)$,

$$\int_I \nu'(s)\,\mathrm{d}s = 2\,\pi\,m = 0,$$

the expression of Theorem 5.32 turns into

$$\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}} \int_I \sin\nu(s)\ \mathrm{d}s - \frac{1-\cos\big(\sqrt{K}\,p\big)}{K} \int_I \kappa_g(s)\ \mathrm{d}s, \tag{5.31}$$

where κ_g is the geodesic curvature of α. From the Gauss–Bonnet formula for α (see Remark 5.15), we have

$$\int_I \kappa_g(s)\ \mathrm{d}s = 2\pi - K\mathcal{A}(\alpha).$$

Substituting this in Equation (5.31), the result is obtained. □

Now, let's translate the ideas above to give a proof of Holditch's theorem in M^K for convex curves where overlapping regions are avoided. The notion of swept out area must be translated into a difference of areas.

Corollary 5.34 (Holditch's theorem in M^K for convex curves)**.** *Let α be a positively oriented convex simple and regular curve in a 2-dimensional manifold of constant curvature K. Suppose a chord of constant length ℓ moves forwards with both endpoints lying in α such that a full revolution (according to the orientation of M^K) is done. A point dividing the chord into two parts of lengths p and q will describe another closed curve γ. If p is admissible and sufficiently small, then*

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = C_{p,q}(K) \cdot \left(1 - \frac{K\mathcal{A}(\alpha)}{2\pi}\right),$$

where

$$C_{p,q}(K) := 4\pi\, \frac{\sin\left(\sqrt{K}\,\frac{p}{2}\right)\sin\left(\sqrt{K}\,\frac{q}{2}\right)}{K\cos\left(\sqrt{K}\,\frac{p+q}{2}\right)}.$$

Proof. First of all, let's see that the convexity of α and a sufficiently small p implies hypothesis $\mathcal{SR}(\alpha, \gamma)$. Note that hypothesis $\mathcal{S}(\alpha)$ holds. Now, in a Holditch setting, since α is convex, we have $b_s(0) = \sin\nu(s) > 0$, so that there exists some $\bar{p} \in [0, \ell]$ such that $b_s(u) > 0$ for all $s \in I$ and $u \in [0, \bar{p}]$. Therefore, $\mathcal{SR}(\alpha, \gamma)$ holds for any Holditch curve γ generated by a chord length $\ell = p + q$ with $p \leq \bar{p}$.

Hence, if α is convex, hypothesis $\mathcal{SR}(\alpha, \gamma)$ holds for any Holditch curve γ generated by a chord length $\ell = p+q$ with $p \leq \bar{p}$ and thus Corollary 5.33 can be used. The convexity of α ensures that the area swept out by the moving chord is equal to the difference

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = \frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}} \int_I \sin\nu(s)\,\mathrm{d}s - \Big(1 - \cos\big(\sqrt{K}\,p\big)\Big)\left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right). \tag{5.32}$$

Now, to compute $\int_I \sin\nu(s)\,\mathrm{d}s$ note that the Holditch curve taking p equal to the length ℓ of the chord is another parameterization of the original curve α. Nevertheless, Corollary 5.33 cannot be used directly for $p = \ell$ since the moving chord may have a point in $[0, \ell]$ where the normal Jacobi field is zero, so that hypothesis $\mathcal{SR}(\alpha, \bar{\alpha})$ would not hold if $\bar{\alpha}$ is the other endpoint of the Holditch chord. Let $\tilde{p} : I \to\,]0, \ell]$ be the length such that for each $s \in I$, $b_s\big(\tilde{p}(s)\big) = 0$. This length function determines, as a generated curve $\tilde{\gamma}$, the envelope curve given by all the positions of the moving chord (see Corollary 5.11). Let

$$F(p) := \frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}} \sin\nu(s) - \frac{1 - \cos\big(\sqrt{K}\,p\big)}{K} \big(\nu'(s) + \kappa_g(s)\big).$$

Thus, we have

$$\mathcal{A}(\alpha) - \mathcal{A}(\tilde{\gamma}) = \int_I \int_0^{\tilde{p}(s)} \sqrt{g(s,u)}\,\mathrm{d}u\,\mathrm{d}s = \int_I \Big(F\big(\tilde{p}(s)\big) - F(0)\Big)\,\mathrm{d}s$$

and also, since $b_s(u) < 0$ for any $u \in\,]\tilde{p}(s), \ell]$, we have $\sqrt{g(s,u)} = -b_s(u)$ and

$$\mathcal{A}(\alpha) - \mathcal{A}(\tilde{\gamma}) = \int_I \int_{\tilde{p}(s)}^{\ell} \sqrt{g(s,u)}\,\mathrm{d}u\,\mathrm{d}s = \int_I \Big(-F(\ell) + F\big(\tilde{p}(s)\big)\Big)\,\mathrm{d}s.$$

Since $F(0) = 0$, we deduce

$$\int_I F(\ell)\,\mathrm{d}s = F(\ell)\,\mathcal{L}(\alpha) = 0.$$

Therefore,

$$F(\ell) = \frac{\sin\big(\sqrt{K}\,\ell\big)}{\sqrt{K}} \sin\nu(s) - \frac{1 - \cos\big(\sqrt{K}\,\ell\big)}{K} \big(\nu'(s) + \kappa_g(s)\big) = 0.$$

Integrating on I,

$$\frac{\sin\big(\sqrt{K}\,\ell\big)}{\sqrt{K}} \int_I \sin\nu(s) - \Big(1 - \cos\big(\sqrt{K}\,\ell\big)\Big)\left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right) = 0$$

and, thus,

$$\int_I \sin\nu(s)\,\mathrm{d}s = \frac{\sqrt{K}}{\sin\big(\sqrt{K}\,\ell\big)} \Big(1 - \cos\big(\sqrt{K}\,\ell\big)\Big)\left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right).$$

Substituting this in Equation (5.32) and simplifying it,

$$\mathcal{A}(\alpha) - \mathcal{A}(\gamma) = \frac{\sin\bigl(\sqrt{K}\,p\bigr) - \sin\bigl(\sqrt{K}\,\ell\bigr) + \sin\bigl(\sqrt{K}\,(\ell - p)\bigr)}{\sin\bigl(\sqrt{K}\,\ell\bigr)} \left(\frac{2\pi}{K} - \mathcal{A}(\alpha)\right).$$

Noticing that

$$\frac{\sin\bigl(\sqrt{K}\,p\bigr) - \sin\bigl(\sqrt{K}\,\ell\bigr) + \sin\bigl(\sqrt{K}\,(\ell - p)\bigr)}{\sin\bigl(\sqrt{K}\,\ell\bigr)} = 2\,\frac{\sin\Bigl(\sqrt{K}\,\frac{p}{2}\Bigr)\,\sin\Bigl(\sqrt{K}\,\frac{\ell-p}{2}\Bigr)}{\cos\Bigl(\sqrt{K}\,\frac{\ell}{2}\Bigr)}$$

and arranging the expression above, the result is obtained. □

5.4.3 The Holditch constant as the area of a closed curve in M^K

The main fact in the statement of Holditch's theorem in the plane is the role of the ellipse: the difference between the area of the initial curve and the area of the Holditch curve is equal to the area of an ellipse which depends only on the lengths p and $q = \ell - p$ in which the chord is divided, but not on the initial curve. On a constant curvature surface the situation is rather different: the difference of areas also depends on the initial curve, but just through its area (Theorem 5.27). Thus, if two curves on a surface have the same area, then the respective differences between their areas and the areas of their Holditch curves are the same.

A natural question that arises in this setting is if the Holditch constant

$$C_{p,q}(K) = 4\pi\,\frac{\sin\Bigl(\sqrt{K}\,\frac{p}{2}\Bigr)\sin\Bigl(\sqrt{K}\,\frac{q}{2}\Bigr)}{K\cos\Bigl(\sqrt{K}\,\frac{p+q}{2}\Bigr)}$$

of Holditch's theorem statement can be realized as the area of a closed curve in M^K. The answer is affirmative and to get that is the objective of this section.

In the planar case, an easy way to make the involved ellipse appear explicitly is to consider a right angle formed by two rays as initial curve, so that the associated Holditch curve is a quarter of the ellipse (this is da Vinci's construction of an ellipse, see Proposition 3.39). Therefore, the case of a right angle in $\mathbb{S}^2(K)$ and $\mathbb{H}^2(K)$ may be considered in order to get the desired curve naturally.

The general quartic curve

First of all, notice that $C_{p,q}(K) > 0$ if $K < 0$. In the case $K > 0$, if $\ell < \frac{\pi}{\sqrt{K}}$, then $C_{p,q}(K) > 0$ also.

Proposition 5.35. *Let $\ell < \frac{\pi}{2\sqrt{K}}$ if $K > 0$. The Holditch curve generated by a chord of length $\ell = p + q$ with its endpoints lying in two different rays forming a right angle in M^K is the intersection with M^K of the quartic curve $\mathbb{Q}_{p,q}(K)$ defined by*

$$\cos^2\!\big(\sqrt{K}\,\ell\big) = \left(1 - \frac{K\sin^2\!\big(\sqrt{K}\,\ell\big)}{\sin^2\!\big(\sqrt{K}\,p\big)}\,y^2\right)\left(1 - \frac{K\sin^2\!\big(\sqrt{K}\,\ell\big)}{\sin^2\!\big(\sqrt{K}\,q\big)}\,z^2\right). \tag{5.33}$$

Proof. The function $\mathbf{x}^K :]0, 2\pi[\times V \to \mathbb{R}^3$ defined by

$$\mathbf{x}^K(u,v) = \left(\frac{\cos\big(\sqrt{K}\,v\big)}{\sqrt{|K|}}, \frac{\sin(u)\sin\big(\sqrt{K}\,v\big)}{\sqrt{K}}, \frac{\cos(u)\sin\big(\sqrt{K}\,v\big)}{\sqrt{K}}\right)$$

is a general parameterization of the sphere $\mathbb{S}^2(K)$ for $K > 0$ and $V = \big]0, \pi/\sqrt{K}\big[$, and also of the hyperboloid $\mathbb{H}^2(K)$ for $K < 0$ and $V =]0, +\infty[$ (see Section 4.2.2).

Consider the rays $\mathbf{x}^K(0,t)$ and $\mathbf{x}^K(\pi/2,t)$, which form a right angle. Thus, if one of the endpoints of the Holditch chord is parameterized by

$$P := P(t) = \mathbf{x}^K(0,t) = \left(\frac{\cos\big(\sqrt{K}\,t\big)}{\sqrt{|K|}}, 0, \frac{\sin\big(\sqrt{K}\,t\big)}{\sqrt{K}}\right), \quad t \in [0,\ell],$$

then the other endpoint is

$$Q := Q(t) = \left(\frac{\cos\big(\sqrt{K}\,\ell\big)}{\sqrt{|K|}\cos\big(\sqrt{K}\,t\big)}, \frac{1}{\sqrt{K}}\sqrt{1 - \frac{\cos^2\big(\sqrt{K}\,\ell\big)}{\cos^2\big(\sqrt{K}\,t\big)}}, 0\right), \quad t \in [0,\ell].$$

By Propositions 4.33 and 4.36, the Holditch curve can be parameterized with the unit speed geodesic that goes from P to Q as

$$\gamma := \gamma(t) = \cos\big(\sqrt{K}\,p\big)\,P - \frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}}\,\frac{P \wedge (P \wedge Q)}{\big\|P \wedge (P \wedge Q)\big\|},$$

that, explicitly, it is

$$\left(\frac{\cos\big(\sqrt{K}\,p\big)\cos\big(\sqrt{K}\,t\big)}{\sqrt{|K|}} + \frac{\cot\big(\sqrt{K}\,\ell\big)\sin\big(\sqrt{K}\,p\big)\sin^2\big(\sqrt{K}\,t\big)}{\sqrt{|K|}\cos\big(\sqrt{K}\,t\big)},\right.$$
$$\left.\frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}\sin\big(\sqrt{K}\,\ell\big)}\sqrt{1 - \frac{\cos^2\big(\sqrt{K}\,\ell\big)}{\cos^2\big(\sqrt{K}\,t\big)}}, \frac{\sin\big(\sqrt{K}\,(\ell - p)\big)\sin\big(\sqrt{K}\,t\big)}{\sqrt{K}\sin\big(\sqrt{K}\,\ell\big)}\right).$$

In order to obtain the implicit equation the parameter t must be removed from the equations

$$z = \frac{\sin\big(\sqrt{K}\,(\ell - p)\big)\sin\big(\sqrt{K}\,t\big)}{\sqrt{K}\sin\big(\sqrt{K}\,\ell\big)}$$

and

$$y = \frac{\sin\left(\sqrt{K}\,p\right)}{\sqrt{K}\sin\left(\sqrt{K}\,\ell\right)}\sqrt{1-\frac{\cos^2\left(\sqrt{K}\,\ell\right)}{\cos^2\left(\sqrt{K}\,t\right)}}.$$

From the first equation we get

$$t = \frac{1}{\sqrt{K}}\arcsin\left(\frac{\sqrt{K}\sin\left(\sqrt{K}\,\ell\right)z}{\sin\left(\sqrt{K}\,(\ell-p)\right)}\right).$$

Substituting in the second equation,

$$y = \frac{\sin\left(\sqrt{K}\,p\right)}{\sqrt{K}\,\sin\left(\sqrt{K}\,\ell\right)}\sqrt{1-\frac{\cos^2\left(\sqrt{K}\,\ell\right)}{1-\frac{K\sin^2\left(\sqrt{K}\,\ell\right)}{\sin^2\left(\sqrt{K}\,(\ell-p)\right)}z^2}}.$$

By squaring and arranging the previous equation the desired result is finally obtained. □

The expression (5.33) of $\mathbb{Q}_{p,q}(K)$ for $K<0$ can be more appropriately written as

$$\cosh^2\left(\sqrt{-K}\,\ell\right) = \left(1+\frac{K\sinh^2\left(\sqrt{-K}\,\ell\right)}{\sinh^2\left(\sqrt{-K}\,p\right)}\,y^2\right)\left(1+\frac{K\sinh^2\left(\sqrt{-K}\,\ell\right)}{\sinh^2\left(\sqrt{-K}\,q\right)}\,z^2\right).$$

Now, let's prove that the closed curve in M^K determined by the quartic $\mathbb{Q}_{p,q}(K)$ has area equal to the Holditch constant $C_{p,q}(K)$.

Theorem 5.36. *Let $\ell < \frac{\pi}{2\sqrt{K}}$ if $K>0$. The area within the quartic curve $\mathbb{Q}_{p,q}(K)$ given by (5.33) on M^K is equal to $C_{p,q}(K)$.*

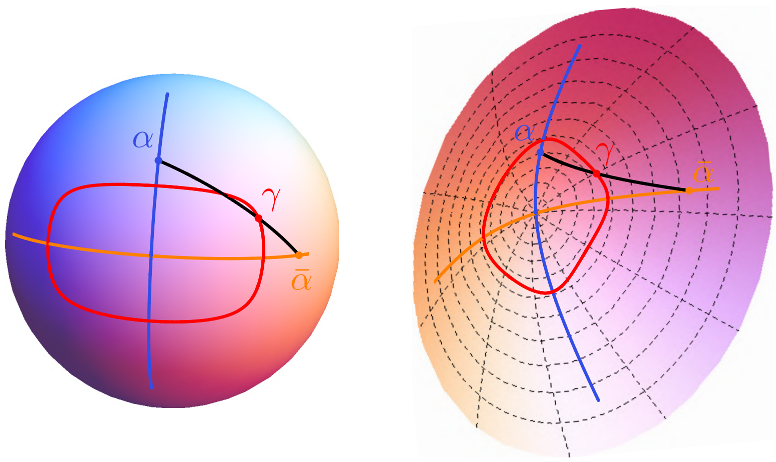

Figure 5.13: The initial curves α and $\bar{\alpha}$ and the Holditch curve γ defined by the quartic curve $\mathbb{Q}_{p,q}(K)$ in the sphere ($K>0$) and in the hyperboloid ($K<0$).

Proof. Let ℓ be the length of the moving chord. Consider two initial curves, α and $\bar{\alpha}$ defined by a double-traced geodesic segment of length $2\,\ell$ forming a right angle at the vertex $\left(1/\sqrt{|K|},\, 0,\, 0\right)$. The Holditch curve, γ, associated with α and $\bar{\alpha}$ and the chord length $\ell = p + q$ is regular and simple (see the proof of Proposition 5.35). See in Figure 5.13 a representation of this setting.

The movement of the moving chord must be done such that the Holditch curve is positively oriented. This implies that a negative full revolution is done ($n = -1$). Although α and $\bar{\alpha}$ are not simple curves, the Gauss–Bonnet theorem can also be applied in this case, so that their areas are $\mathcal{A}(\alpha) = \mathcal{A}(\bar{\alpha}) = 0$ and Holditch's theorem for two initial curves—Theorem 5.25—produces: $\mathcal{A}(\gamma) = C_{p,q}(K)$. $\square$

See in Figures 5.14 and 5.15 the Holditch curve of Proposition 5.35 found by the intersection of M^K with the quartic curve $\mathbb{Q}_{p,q}(K)$ together with its projection onto the yz-plane.

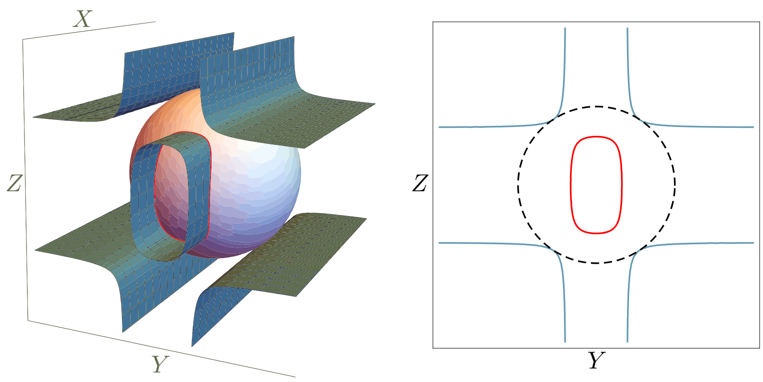

Figure 5.14: On the left, for $K > 0$, $\ell = 1$ and $p = 1/3$, the curve whose area is $C_{p,q}(K)$ as the intersection of the quartic curve $\mathbb{Q}_{p,q}(K)$ with the sphere $\mathbb{S}^2(K)$. On the right, the yz-projection of this setting.

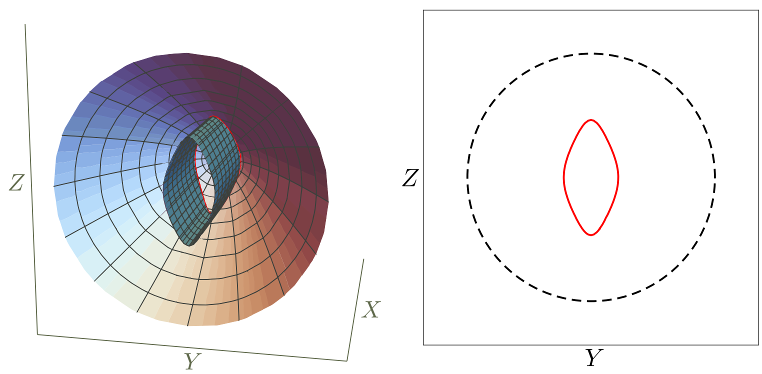

Figure 5.15: On the left, for $K < 0$, $\ell = 1$ and $p = 1/3$, the curve whose area is $C_{p,q}(K)$ as the intersection of the quartic curve $\mathbb{Q}_{p,q}(K)$ with the hyperboloid $\mathbb{H}^2(K)$. On the right, the yz-projection of this setting.

Relation with cruciform curves

Quartic plane curves with implicit equations of the kind

$$x^2y^2 - b^2x^2 - a^2y^2 = 0 \tag{5.34}$$

are called *cruciform curves* or *cross curves* (see [52]). Notice that if $x, y \neq 0$, then Equation (5.34) is equivalent to

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1.$$

Thus, cruciform curves are related to ellipses by the transformation $(x, y) \mapsto \left(\frac{1}{x}, \frac{1}{y}\right)$.

Given a and b either real or purely imaginary numbers, define $h_{a,b} : \mathbb{R}^2 \to \mathbb{R}$ by

$$h_{a,b}(x, y) := x^2y^2 - b^2x^2 - a^2y^2.$$

Cruciform curves are the 0-level set of $h_{a,b}$. A simple computation leads to the following lemma.

Lemma 5.37. *Let a, b and c be either real or purely imaginary numbers. The equation $x^2y^2 - b^2x^2 - a^2y^2 = -c^2$ is the same as*

$$\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right) = \frac{-c^2}{a^2b^2} + 1.$$

Note that if we take purely imaginary numbers $\mathbf{i}\,a$, $\mathbf{i}\,b$ and $\mathbf{i}\,c$ with $a, b, c \in \mathbb{R}$ in the previous lemma, we get that the equation $x^2y^2 + b^2x^2 + a^2y^2 = c^2$ is the same as

$$\left(1 + \frac{x^2}{a^2}\right)\left(1 + \frac{y^2}{b^2}\right) = \frac{c^2}{a^2b^2} + 1.$$

Thus, from Lemma 5.37, Equation (5.33) can also be written as

$$y^2\,z^2 - \frac{\sin^2\left(\sqrt{K}\,q\right)}{K\sin^2\left(\sqrt{K}\,\ell\right)}\,y^2 - \frac{\sin^2\left(\sqrt{K}\,p\right)}{K\sin^2\left(\sqrt{K}\,\ell\right)}\,z^2 = -\frac{\sin^2\left(\sqrt{K}\,p\right)\sin^2\left(\sqrt{K}\,q\right)}{K^2\sin^2\left(\sqrt{K}\,\ell\right)}.$$

Again, if $K < 0$, this equation is more appropriately written as

$$\begin{aligned} y^2\,z^2 + \frac{\sinh^2\left(\sqrt{-K}\,q\right)}{K\sinh^2\left(\sqrt{-K}\,\ell\right)}\,y^2 + \frac{\sinh^2\left(\sqrt{-K}\,p\right)}{K\sinh^2\left(\sqrt{-K}\,\ell\right)}\,z^2 & \\ = \frac{\sinh^2\left(\sqrt{-K}\,p\right)\sinh^2\left(\sqrt{-K}\,q\right)}{K^2\sinh^2\left(\sqrt{-K}\,\ell\right)}&. \end{aligned}$$

The relation between the quartic $\mathbb{Q}_{p,q}(K)$ and the function $h_{a,b}$ follows immediately.

Proposition 5.38. *The yz-projection of the quartic curve $\mathbb{Q}_{p,q}(K)$ is the connected component (inside the disk of radius $1/\sqrt{K}$ if $K > 0$) of the $-c^2$-level set of the function $h_{a,b}$, where*

$$a = \frac{\sin\big(\sqrt{K}\,p\big)}{\sqrt{K}\sin\big(\sqrt{K}\,\ell\big)}, \qquad b = \frac{\sin\big(\sqrt{K}\,(\ell - p)\big)}{\sqrt{K}\sin\big(\sqrt{K}\,\ell\big)}$$

$$\text{and} \quad c = \frac{\sin\big(\sqrt{K}\,p\big)\sin\big(\sqrt{K}\,(\ell - p)\big)}{|K|\sin\big(\sqrt{K}\,\ell\big)}.$$

With the values of Proposition 5.38 for $K > 0$, the 0-level set of $h_{a,b}$ consists in a cross shape. Nevertheless, if the right hand member of the implicit equation of the cruciform curve is substituted by a negative number, as happens in the curve $\mathbb{Q}_{p,q}(K)$, then a closed piece appears inside the circle centered at the origin and of radius $1/\sqrt{K}$, while maintaining its cruciform shape outside (see Figure 5.16).

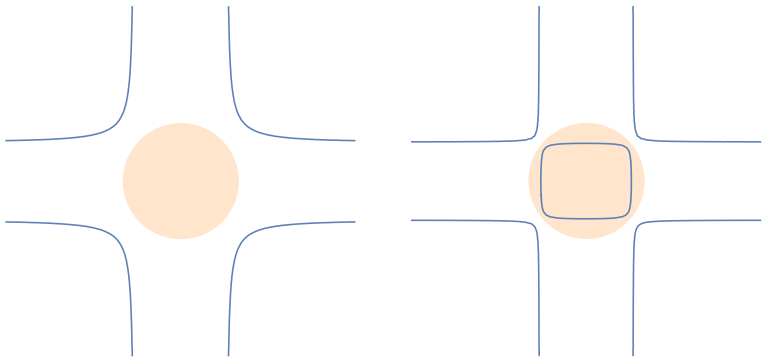

Figure 5.16: Example with $\ell = 1.3$ and $p = 0.8$, taking the values of Proposition 5.38, in the sphere $\mathbb{S}^2(1)$. On the left, the cruciform curve computed as the 0-level set of $h_{a,b}$. On the right, the new curve, projection of $\mathbb{Q}_{p,q}(1)$, as the connected component of the $-c^2$-level set of $h_{a,b}$.

In the hyperbolic case, if a, b and c are purely imaginary numbers, then the quartic curve

$$y^2z^2 - b^2y^2 - a^2z^2 = -c^2$$

does not have a real cross shape. In fact, the 0-level set of $h_{a,b}$ in this case is a single point. For the values of Proposition 5.38, which define the curve $\mathbb{Q}_{p,q}(K)$, the right hand member of that implicit equation is a positive number. In this case, the shape of the curve with such values does not necessarily lies inside the disk centered at the origin and of radius $1/\sqrt{|K|}$ (see Figure 5.17).

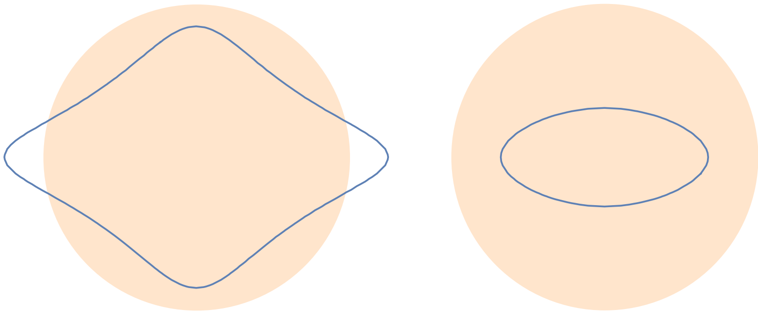

Figure 5.17: Example, with the values of Proposition 5.38, in the hyperboloid $\mathbb{H}^2(-1)$. On the left, the $-c^2$-level set of $h_{a,b}$ for $\ell = 1.7$ and $p = 1.1$. On the right, the same for $\ell = 0.5$ and $p = 0.4$. The unit disk is also plotted.

5.4.4 Characterizing the manifold through Holditch's and Barbier's theorem

The objective of this section is to present some results which are, somehow, a converse of Holditch's and Barbier's theorem in constant curvature surfaces. That is, if a Holditch or a Barbier's type formula works in a 2-dimensional manifold M, then the geometry of M is determined by its Gauss curvature. Indeed, this is not entirely surprising since the area or the length of a geodesic circle determines locally the Gauss curvature of the manifold. Recall also that Minding's theorem states that all surfaces with the same constant curvature are locally isometric.

Definition 5.39 (Geodesic circle)**.** The *geodesic circle of radius r centered at $p \in M$* is defined as the set of points which are at a distance equal to r from p.

The main fact is the expression as power series for the area and the length of a geodesic circle (see page 292 of [20], [33] or page 229 of [51]).

Theorem 5.40. *Let M be a 2-dimensional Riemannian manifold. The area of a geodesic circle α centered at $p \in M$ and with radius r has the following power series expansion in r:*

$$\mathcal{A}(\alpha) = \pi\, r^2 - \frac{1}{12} K\, \pi\, r^4 + \mathcal{O}\big(r^6\big). \tag{5.35}$$

Moreover, the power series expansion in r for the length of α is

$$\mathcal{L}(\alpha) = 2\,\pi\, r - \frac{1}{3} K\, \pi\, r^3 + \mathcal{O}\big(r^5\big). \tag{5.36}$$

The previous power series allow us to recover the Gauss curvature as written below (see [6] and pages 147–148 of [87]).

Theorem 5.41 (Bertrand, Puiseux and Diguet, 1848). *Let M be a Riemannian manifold of dimension 2. Let α be the geodesic circle of radius r centered at $p \in M$. Then*

$$K(p) = \lim_{r \to 0} \frac{\pi r^2 - \mathcal{A}(\alpha)}{\pi\, r^4/12}$$

and

$$K(p) = \lim_{r \to 0} \frac{2\pi r - \mathcal{L}(\alpha)}{\pi\, r^3/3}.$$

Next, the main results are given. The reader can easily notice that with the same procedure presented below, a new proof of Theorem 5.40 for a constant curvature manifold can also be given.

Characterizing the manifold through Holditch's theorem

First, let's see that the classical Holditch formula only works in surfaces which are locally isometric to a plane.

Theorem 5.42. *Let M be a 2-dimensional Riemannian manifold M embedded in $\mathbb{R}^3$. If for every closed curve α,*

$$\mathcal{A}(\alpha) - \mathcal{A}\big(H_\alpha\big) = \pi\, p\, q,$$

with H_α being any Holditch curve generated by chord of constant length $p+q$, then M is locally isometric to a plane.

Proof. Consider a geodesic circle of radius r, for $r > 0$, centered at some $m \in M$. The midpoint Holditch curve generated by a chord of length $2\,r$ reduces to a single point. On the one hand, by hypothesis, the Holditch area is equal to $\pi\, r^2$. On the other hand, the Holditch area is given by Equation (5.35) of Theorem 5.40. Since both areas must agree, then necessarily $K = 0$. $\square$

With the same idea, the previous result can be extended to characterize non-zero constant curvature surfaces.

Theorem 5.43. *Let M be a 2-dimensional Riemannian manifold M embedded in $\mathbb{R}^3$ of Gauss curvature K. If for every closed curve α and any Holditch curve of α, H_α, determined by a chord length $p+q$, it holds*

$$\mathcal{A}(\alpha) - \mathcal{A}\big(H_\alpha\big) = C_{p,q}(K_0) \cdot \left(1 - \frac{K\mathcal{A}(\alpha)}{2\pi}\right),$$

where

$$C_{p,q}(K_0) = 4\pi\, \frac{\sin\big(\sqrt{K_0}\,\frac{p}{2}\big)\sin\big(\sqrt{K_0}\,\frac{q}{2}\big)}{K_0\cos\big(\sqrt{K_0}\,\frac{p+q}{2}\big)}.$$

with K_0 being a non-zero constant, then necessarily $K = K_0$, so M is of constant Gauss curvature.

Proof. Let α be a geodesic circle of radius r and consider the midpoint Holditch curve generated by a chord of length $2\,r$. The expansion of $C_{r,r}(K_0)$ by power series in r is

$$C_{r,r}(K_0) = \pi\, r^2 + \frac{5}{12}\, K_0\, \pi\, r^4 + \mathcal{O}\big(r^6\big).$$

Hence, using Equation (5.35) of Theorem 5.40, we get

$$C_{r,r}(K_0) \cdot \left(1 - \frac{K_0\, \mathcal{A}(\alpha)}{2\pi}\right) = \pi\, r^2 - \frac{1}{12}\, K_0\, \pi\, r^4 + \mathcal{O}\big(r^6\big).$$

Since Holditch's formula holds, then it must be equal to Equation (5.35) of Theorem 5.40, so $K = K_0$. □

Characterizing the manifold through Barbier's theorem

Similarly as above, the classical Barbier formula only works in surfaces which are locally isometric to a plane.

Theorem 5.44. *Let M be a 2-dimensional Riemannian manifold M embedded in $\mathbb{R}^3$. If for every curve α of constant width ℓ,*

$$\mathcal{L}(\alpha) = \pi\, \ell,$$

then M is locally isometric to a plane.

Proof. By hypothesis, a geodesic circle α of radius r has length

$$\mathcal{L}(\alpha) = 2\, \pi\, r,$$

since it is of constant width $2\,r$. Because of Equation 5.36 of Theorem 5.40, necessarily $K = 0$. □

Again, this result can be extended to characterize constant curvature surfaces.

Theorem 5.45. *Let M be a 2-dimensional Riemannian manifold M embedded in $\mathbb{R}^3$ of Gauss curvature K. If for every curve α of constant width ℓ, it holds*

$$\mathcal{L}(\alpha) = \frac{2\pi - K_0\, \mathcal{A}(\alpha)}{\sqrt{K_0}}\, \tan\!\Big(\sqrt{K_0}\, \frac{r}{2}\Big), \tag{5.37}$$

with K_0 being a non-zero constant, then necessarily $K = K_0$, so M is of constant Gauss curvature.

Proof. We can expand by power series in r the right hand side of the expression (5.37) with Equation (5.35)—for the curvature K—of Theorem 5.40, which yields

$$\mathcal{L}(\alpha) = 2\, \pi\, r - \frac{1}{3}\, K_0\, \pi\, r^3 + \mathcal{O}\big(r^5\big).$$

Now, comparing this with Equation (5.36) of Theorem 5.40, we deduce that $K = K_0$. □

Remark 5.46. In Theorems 5.42, 5.43, 5.44 and 5.45, the hypotheses have been written to show the idea of being a "converse" of Holditch and Barbier's theorem although, clearly, only geodesic circles are needed to complete their proofs.

Chapter 6

Holditch's theorem in 3D space

Some generalizations of Holditch's theorem have been studied in previous chapters, but none of these follows, regardless if the curve lies in a surface or not, the same natural construction of the plane with a straight chord but done for a space curve. In this chapter, the notion of Holditch surface is defined, some properties of these surfaces are proved and they are used to generalize Holditch's theorem for straight moving chords in closed space curves. Moreover, an approximation for the area of interest is given. Finally, it is shown that the only minimal non-planar Holditch surface is the helicoid.

6.1 Introduction to the problem

We have studied in previous chapters generalizations of Holditch's theorem in 2-dimensional constant curvature manifolds, taking geodesic chords and working with surface areas.

When the initial curve is a non-planar closed curve in the space, the generation of a new curve by the same procedure as in the plane is still possible (see Figure 6.1). Therefore, a natural problem that arises is to find an analogue of Holditch's theorem for space curves, that is, a result relating lengths on the moving chord with the area of some surface between the initial curve and the generated one.

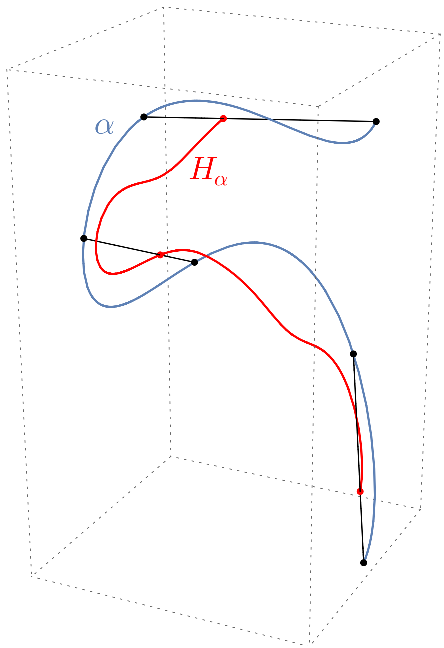

Figure 6.1: The 1/3-Holditch curve of a space curve α generated by a straight chord of constant length.

The main problem is to determine from which surface one has to compute the area. It must be done in such a way that it is naturally defined and such that its area agrees with Holditch's theorem if the initial curve is planar. In a paper by Arnol'd, [2], dealing with caustic curves on the sphere, the author considers the notion of wave front, this is, a family of curves associated with a parameter. Following that idea, a possible candidate surface to compute the Holditch area can be defined.

Given a value $p \in [0, 1]$, our candidate surface is the wave front made by all the Holditch curves defined by a point which divides a chord of length ℓ into two pieces of lengths $p\,\ell$ and $(1-p)\,\ell$. The parameter of the wave front is the length ℓ.

Throughout this chapter, $\alpha : I \to \mathbb{R}^n$ will be a planar ($n = 2$) or a space ($n = 3$) regular curve parameterized by arc-length. This chapter is based on the work [66].

6.2 Definition of Holditch surfaces

Henceforth, the value $p \in [0,1]$ will determine the ratio $p : q$ in which a chord of constant length L is divided into two parts ($q = 1 - p$).

As in the plane, if the chosen length of the chord is long enough, retrograde motion may appear. We will focus on chords of constant length L such that retrograde movements do not appear. In such a case, as in the plane (see Section 3.1.1), recall that the p-Holditch curve can be parameterized as

$$H_\alpha(s) = (1-p)\,\alpha(s) + p\,\alpha\big(f(s)\big),$$

where $f : J_L \to f(J_L)$ is the Holditch function. Since f depends on α but also on L, write explicitly the dependence on the chord length L by naming it $f(s, L)$.

So, given a fixed p, a natural parametric surface can be defined by varying the length of the moving chord. It is formed by all the generated p-Holditch curves of α for each chord length ℓ up to some higher fixed value L. That parametric surface, say $\mathbf{h}^p : I \times]0, L[\to \mathbb{R}^n$, will be called the *$p$-Holditch surface of α up to a length L* and it is defined by

$$\mathbf{h}^p(s,\ell) = (1-p)\,\alpha(s) + p\,\alpha\big(f(s,\ell)\big). \tag{6.1}$$

Coordinate curves $s \mapsto \mathbf{h}^p(s, \ell_0)$, with ℓ_0 constant, are Holditch curves of α.

Note that we can reparameterize the surface by the change $u = f(s,\ell) - s$ to simplify its expression as we state in Definition 6.1. Define

$$m(s, L) := f(s, L) - s.$$

Definition 6.1 (Holditch surface)**.** Let $L > 0$ be a chord length such that retrograde motion is avoided for α. The *p-Holditch surface of α up to L* is the parametric surface $\mathbf{x}^p : U_L \to \mathbb{R}^n$ defined by

$$\mathbf{x}^p(s,u) = (1-p)\,\alpha(s) + p\,\alpha(s+u). \tag{6.2}$$

where $U_L = \big\{(s,u) \ : \ s \in I,\ u \in]0, m(s,L)[\big\}$. If there is no ambiguity in α, p and L, we will call it simply the *Holditch surface.*

Now, coordinate curves $s \mapsto \mathbf{x}^p(s, u_0)$, with u_0 constant, are no longer Holditch curves. Also, notice that

$$\lim_{u\to 0} \mathbf{x}^p(s,u) = \alpha(s).$$

The definition of Holditch surface works for any curve, but we will focus later on closed curves in order to generalize Holditch's theorem for space closed curves. Look at Figure 6.2 to see some examples of Holditch surfaces for closed curves.

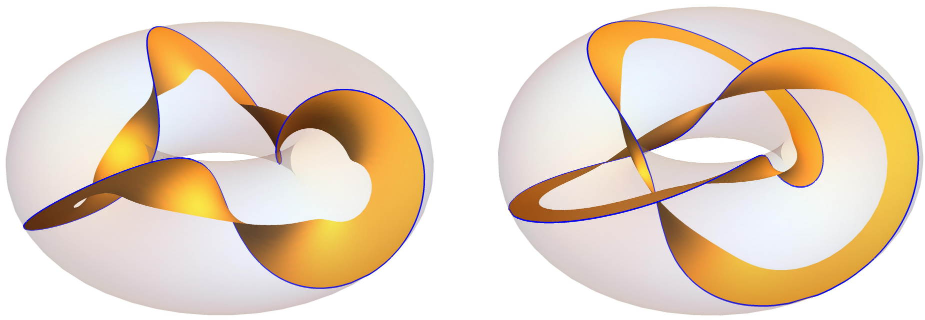

Figure 6.2: Two curves in a torus and their 1/2-Holditch surfaces for $L = 5/2$.

Similarly as in Section 3.1.3, define $\phi(s, u)$ as the angle between $\mathbf{t}(s)$ and $\mathbf{t}(s + u)$. It is easy to show that if α has non-vanishing curvature at $s \in I$, then there exists some value $\tilde{L}$ such that for lengths $0 < L < \tilde{L}$, the angle $\phi(s, u)$ is not zero for $u \in]0, m(s, L)[$. To ensure the regularity of Holditch surfaces, this restriction on the chord length will be set additionally to the condition of avoiding retrograde movements.

Definition 6.2 (Length admissible for regularity)**.** Given the curve α, we will say that a length L is *admissible for regularity with regard to* α if there is no retrograde motion on α for a chord length L and if $\phi(s, u) \neq 0$ for all $(s, u) \in U_L$. If there is no ambiguity, we will say simply a *length admissible for regularity.*

Proposition 6.3. *For any* $p \in [0, 1]$*, the* p*-Holditch surface of* α *up to an admissible for regularity length is regular in its domain.*

Proof. The derivatives of (6.2) with respect to s and u are

$$\mathbf{x}_s^p(s, u) = (1 - p)\,\mathbf{t}(s) + p\,\mathbf{t}(s + u),$$
$$\mathbf{x}_u^p(s, u) = p\,\mathbf{t}(s + u).$$

Therefore, their cross product is

$$(\mathbf{x}_s^p \wedge \mathbf{x}_u^p)(s, u) = (1 - p)\,p\,\mathbf{t}(s) \wedge \mathbf{t}(s + u)$$

and its norm:

$$\big\|(\mathbf{x}_s^p \wedge \mathbf{x}_u^p)(s, u)\big\| = (1 - p)\,p\,\sin\phi(s, u). \tag{6.3}$$

Since $\phi(s, u) \neq 0$ for the length L and $u \neq 0$, we have that regularity is ensured. □

6.3 Some properties of Holditch surfaces

With the previous calculations we have the following.

Proposition 6.4. *The normal vector of a p-Holditch surface of a curve up to an admissible for regularity length is given by*

$$\mathbf{N}^{\mathbf{x}^p}(s,u) = \frac{(\mathbf{x}_s^p \wedge \mathbf{x}_u^p)(s,u)}{\big\|(\mathbf{x}_s^p \wedge \mathbf{x}_u^p)(s,u)\big\|} = \frac{\mathbf{t}(s) \wedge \mathbf{t}(s+u)}{\sin\phi(s,u)},$$

which is independent of p*.*

Notice that for different p values one gets different surfaces, but all of them start at the initial curve and has as tangent plane there the osculating plane of the initial curve.

Proposition 6.5. *Let* $L > 0$ *be a length admissible for regularity. The limit of the tangent planes of the p-Holditch surfaces of* α *up to* ℓ *at some point, for lengths* $0 < \ell < L$ *tending to zero, is the osculating plane of* α *at the same point.*

Proof. It is sufficient to show that

$$\lim_{u\to 0} \mathbf{N}^{\mathbf{x}^p}(s,u) = \mathbf{b}(s).$$

That can be seen as a limiting case from the point of view of discrete geometry, since $\mathbf{N}^{\mathbf{x}^p}$ represents a discrete version of the binormal vector of α. Another possibility is to compute the limit using the L'Hôpital's rule. Indeed,

$$\lim_{u\to 0} \mathbf{N}^{\mathbf{x}^p}(s,u) = \lim_{u\to 0} \frac{\kappa(s+u)\,\mathbf{t}(s) \wedge \mathbf{n}(s+u)}{\cos\phi(s,u)\,\phi_u(s,u)} = \mathbf{b}(s),$$

since $\phi(s,0) = 0$ and $\phi_u(s,0) = \kappa(s)$ (see Lemma 6.12 below). □

With the coefficients of the first and second fundamental forms we can get the Gauss and the mean curvature of Holditch surfaces.

Proposition 6.6. *For any* $p \in [0,1]$*, the p-Holditch surface of* α *up to an admissible for regularity length has Gauss curvature*

$$K(s,u) = -\frac{\kappa(s)\,\kappa(s+u)}{p\,q\sin^4\phi(s,u)} \big\langle \mathbf{t}(s+u), \mathbf{b}(s) \big\rangle \big\langle \mathbf{b}(s+u), \mathbf{t}(s) \big\rangle$$

and mean curvature

$$H(s,u) = \frac{1}{2\sin^3\phi(s,u)} \left(\frac{\kappa(s+u)}{p} \big\langle \mathbf{b}(s+u), \mathbf{t}(s) \big\rangle - \frac{\kappa(s)}{q} \big\langle \mathbf{t}(s+u), \mathbf{b}(s) \big\rangle \right).$$

Proof. Let $\mathbf{x}^p$ be the p-Holditch surface of α up to L as in (6.2). The coefficients of the first fundamental form are given by

$$\begin{aligned} E(s,u) &= p^2 + q^2 + 2\,p\,q\cos\phi(s,u), \\ F(s,u) &= p^2 + p\,q\,\cos\phi(s,u), \\ G(s,u) &= p^2. \end{aligned}$$

Therefore,

$$\big(E\,G - F^2\big)(s,u) = p^2\,q^2\sin^2\phi(s,u). \tag{6.4}$$

Now, since

$$\mathbf{x}^p_{ss}(s,u) = q\,\kappa(s)\,\mathbf{n}(s) + p\,\kappa(s+u)\,\mathbf{n}(s+u),$$

and

$$\mathbf{x}^p_{su}(s,u) = \mathbf{x}^p_{uu}(s,u) = p\,\kappa(s+u)\,\mathbf{n}(s+u),$$

by using Proposition 6.4 the coefficients of the second fundamental form are easily obtained:

$$\begin{aligned} e(s,u) &= \frac{p\,\kappa(s+u)}{\sin\phi(s,u)}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle - \frac{q\,\kappa(s)}{\sin\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle, \\ f(s,u) &= \frac{p\,\kappa(s+u)}{\sin\phi(s,u)}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle = g(s,u). \end{aligned}$$

Hence,

$$\big(e\,g - f^2\big)(s,u) = -\frac{p\,q\,\kappa(s)\,\kappa(s+u)}{\sin^2\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle. \tag{6.5}$$

The expression of the Gauss curvature of the statement is obtained by dividing (6.5) by (6.4). Also, dividing

$$\begin{aligned} &(e\,G - 2\,f\,F + g\,E)(s,u) \\ &\quad = \frac{p\,q^2\,\kappa(s+u)}{\sin\phi(s,u)}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle - \frac{p^2\,q\,\kappa(s)}{\sin\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle \end{aligned}$$

by (6.4) and multiplying by 1/2, the expression of the mean curvature is deduced. □

Immediately, from (6.3) we get the expression for the area of Holditch surfaces.

Proposition 6.7. *Given $p \in [0,1]$, the area of the p-Holditch surface of α up to an admissible for regularity length $L > 0$ is*

$$A_H(p,L) = p\,(1-p)\int_I \int_0^{m(s,L)} \sin\phi(s,u)\;\mathrm{d}u\;\mathrm{d}s.$$

6.4 Holditch surfaces of planar curves

The aim of this section is to give a new proof of Holditch's theorem in the plane by using Holditch surfaces. The first point is to see that the Holditch surface of a planar curve describes the Holditch region, i.e., the region determined by the initial curve and its Holditch curve.

Proposition 6.8. *Any p-Holditch surface of a planar curve α up to some admissible for regularity length $L > 0$ is the planar region determined by α and the p-Holditch curve of α for the length L.*

Proof. The p-Holditch surface up to L consists on all the p-Holditch curves for lengths ℓ from 0 to L. We know that the p-Holditch curves of a planar curve are also planar in the same plane and that they are nested and do not intersect to each other given two different lengths. That is consequence of the regularity of the Holditch surface (Proposition 6.3)—its coordinate lines with (6.1) are Holditch curves—and of being the change $f(s, \ell) = s + u$ injective. Therefore, the region determined by the Holditch surface is the region between α and the p-Holditch curve for the length L. □

By Proposition 6.8, overlapping area portions will not appear in planar Holditch surfaces for admissible for regularity lengths. Thus, the area of the region determined by a Holditch surface in the plane agrees with the notion of Holditch area. With that, a new proof of the classical Holditch theorem is possible using Holditch surfaces. Before that, recall Corollary 3.24.

Lemma 6.9. *Suppose the angle ϕ to be defined by a chord of an admissible for regularity length. If α is a strictly convex planar curve, then*

$$\phi(s, u) = \int_s^{s+u} \kappa(t)\, \mathrm{d}t,$$

where κ is the curvature function of α.

Proof. Let $\sigma : I \to \mathbb{R}$ be the oriented angle function from the OX axis to $\mathbf{t}(s)$. By definition of ϕ, we have that

$$\phi(s, u) = \sigma(s + u) - \sigma(s) = \int_s^{s+u} \sigma'(t)\, \mathrm{d}t.$$

The result is obtained since $\sigma' = \kappa$. □

The next theorem shows the proof of the classical Holditch theorem by describing the Holditch region with a Holditch surface.

Theorem 6.10. *The p-Holditch surface of a closed strictly convex planar curve up to an admissible for regularity length $L > 0$ has planar area equal to the Holditch area: $\pi\, p\, (1 - p)\, L^2$.*

Proof. According to Proposition 6.8, the p-Holditch surface will be the planar region between the initial curve α and its p-Holditch curve for the length L. That region will have an area

$$A_H(p,L) = p\,(1-p)\int_I \int_0^{m(s,L)} \sin\phi(s,u)\;\mathrm{d}u\;\mathrm{d}s$$

by Proposition 6.7.

Recall that given the curvature κ of a planar curve $\big(x(s), y(s)\big)$, we can recover

$$x'(s) = \cos\int \kappa(s)\;\mathrm{d}s, \quad y'(s) = \sin\int \kappa(s)\;\mathrm{d}s,$$

up to a rotation (given by the integration constant in the chosen primitive of κ). Therefore, using Lemma 6.9,

$$\begin{aligned}
\sin\phi(s,u) &= \sin\int_s^{s+u} \kappa(t)\;\mathrm{d}t = \sin\left(\int_0^{s+u}\kappa(t)\;\mathrm{d}t - \int_0^s \kappa(t)\;\mathrm{d}t\right)\\
&= \sin\int_0^{s+u}\kappa(t)\;\mathrm{d}t\,\cos\int_0^s\kappa(t)\;\mathrm{d}t - \cos\int_0^{s+u}\kappa(t)\;\mathrm{d}t\,\sin\int_0^s\kappa(t)\;\mathrm{d}t\\
&= y'(s+u)\,x'(s) - x'(s+u)\,y'(s).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^{m(s,L)} \sin\phi(s,u)\;\mathrm{d}u\\
&\qquad = x'(s)\int_0^{m(s,L)} y'(s+u)\;\mathrm{d}u - y'(s)\int_0^{m(s,L)} x'(s+u)\;\mathrm{d}u\\
&\qquad = x'(s)\,y\big(s+m(s,L)\big) - y'(s)\,x\big(s+m(s,L)\big)\\
&\qquad\qquad - x'(s)y(s) + y'(s)\,x(s).
\end{aligned}$$

The area of α is

$$A = \int_I x(s)\,y'(s)\;\mathrm{d}s = -\int_I x'(s)\,y(s)\;\mathrm{d}s.$$

Hence,

$$\frac{A_H(p,L)}{p\,(1-p)} = 2\,A + \int_I \Big(x'(s)\,y\big(s+m(s,L)\big) - y'(s)\,x\big(s+m(s,L)\big)\Big)\;\mathrm{d}s. \tag{6.6}$$

We know that the curve $\alpha\big(s+m(s,L)\big) - \alpha(s)$ is a circle of radius L. Its area, $\pi\,L^2$, can be computed with

$$\begin{aligned}
&\int_I \Big(x\big(s+m(s,L)\big) - x(s)\Big)\Big(\Big(y\big(s+m(s,L)\big)\Big)' - y'(s)\Big)\;\mathrm{d}s\\
&\quad = \int_I x\big(s+m(s,L)\big)\Big(y\big(s+m(s,L)\big)\Big)'\;\mathrm{d}s - \int_I x\big(s+m(s,L)\big)y'(s)\;\mathrm{d}s\\
&\qquad - \int_I x(s)\Big(y\big(s+m(s,L)\big)\Big)'\;\mathrm{d}s + \int_I x(s)\,y'(s)\;\mathrm{d}s\\
&\quad = 2\,A + \int_I \Big(x'(s)\,y\big(s+m(s,L)\big) - y'(s)\,x\big(s+m(s,L)\big)\Big)\;\mathrm{d}s.
\end{aligned}$$

Finally, using this on (6.6) we get

$$A_H(p, L) = \pi\, p\, (1 - p)\, L^2,$$

which is the Holditch area. □

In Figure 6.3 two examples of planar curves are represented. The planar region determined by the Holditch surface coincides with the Holditch region.

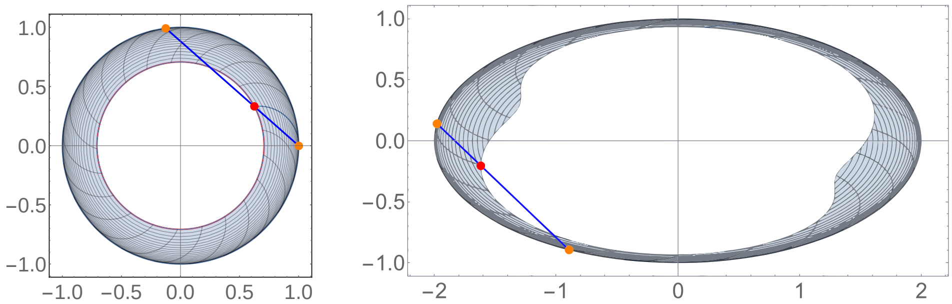

Figure 6.3: The 1/3-Holditch surfaces for $L = 3/2$ in a unit circle and in an ellipse with semi-axes 2 and 1. The region determined by such a surface has area equal to the Holditch area.

6.5 Holditch surfaces of space curves

In this section we will deal with space curves. The immediate result which generalizes Holditch's theorem to the space is stated next. It follows directly from Proposition 6.7.

Theorem 6.11. *Given $p \in [0,1]$, if $A_H(p,L)$ is the area of the p-Holditch surface of a curve α up to an admissible for regularity length $L > 0$, then the quantity*

$$\frac{A_H(p,L)}{p\,(1-p)}$$

is independent of p.

In the planar case, the quotient of Theorem 6.11 is independent of p and α. In the space case, the situation is different, since although it does not depend on p, the dependence on α is not avoided. In fact, in the known generalizations of Holditch's theorem for constant curvature surfaces, the dependence of α in the full Holditch formula is not avoided either (see Chapter 5).

The following lemma is just a matter of computation. It will be used later to find an approximation for the area of a Holditch surface in the space case.

Lemma 6.12. *The three first derivatives of $m(s,L)$ with respect to L at $L = 0$ are*

$$m_L(s,0) = 1, \quad m_{LL}(s,0) = 0, \quad m_{LLL}(s,0) = \frac{1}{4}\,\kappa^2(s),$$

and the same of $\phi(s,u)$ with respect to u at $u = 0$ are

$$\phi_u(s,0) = \kappa(s), \quad \phi_{uu}(s,0) = \kappa'(s), \quad \phi_{uuu}(s,0) = \kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s),$$

where κ and τ are the curvature and torsion, respectively, of a curve α with non-vanishing curvature.

Proof. We have two conditions from which we can compute the derivatives of m with respect to L and of ϕ with respect to u. These are

$$\Big\|\alpha\big(s + m(s,L)\big) - \alpha(s)\Big\|^2 = L^2 \tag{6.7}$$

and

$$\cos\phi(s,u) = \big\langle \mathbf{t}(s), \mathbf{t}(s+u)\big\rangle. \tag{6.8}$$

Let's begin with the derivatives of m. Differentiating (6.7) with respect to L, we get

$$m_L(s,L)\,\Big\langle \mathbf{t}\big(s + m(s,L)\big), \alpha\big(s + m(s,L)\big) - \alpha(s)\Big\rangle = L.$$

Differentiating again,

$$\begin{aligned}&m_{LL}\left\langle \mathbf{t}(s+m), \alpha(s+m)-\alpha(s)\right\rangle\\&\qquad + m_L\Big(\kappa(s+m)\, m_L \left\langle \mathbf{n}(s+m), \alpha(s+m)-\alpha(s)\right\rangle + m_L\Big) = 1.\end{aligned}$$

Evaluating at $L=0$, since $m(s,0)=0$, we find that

$$m_L(s,0)=1.$$

The third derivative of (6.7) yields

$$\begin{aligned}&\big(m_{LLL}-\kappa^2(s+m)\, m_L^3\big)\left\langle \mathbf{t}(s+m), \alpha(s+m)-\alpha(s)\right\rangle\\&+\big(3\, m_L\, m_{LL}\,\kappa(s+m)+m_L^3\,\kappa'(s+m)\big)\left\langle \mathbf{n}(s+m), \alpha(s+m)-\alpha(s)\right\rangle\\&- m_L^3\,\kappa(s+m)\,\tau(s+m)\left\langle \mathbf{b}(s+m), \alpha(s+m)-\alpha(s)\right\rangle + 3\, m_L\, m_{LL} = 0.\end{aligned}$$

Evaluating at $L=0$, since $m_L(s,0)=1$, we get $m_{LL}(s,0)=0$.

Hence, the fourth derivative evaluated at $L=0$ gives

$$4\, m_{LLL}(s,0)-\kappa^2(s)=0,$$

so that $m_{LLL}(s,0)=\frac{1}{4}\,\kappa^2(s)$.

Now, let's compute the derivatives of ϕ. Differentiating (6.8) with respect to u, we get

$$-\sin\phi(s,u)\,\phi_u(s,u)=\kappa(s+u)\left\langle \mathbf{t}(s), \mathbf{n}(s+u)\right\rangle.$$

Differentiating again,

$$\begin{aligned}&\cos\phi(s,u)\left(\kappa^2(s+u)-\phi_u^2(s,u)\right)-\sin\phi(s,u)\,\phi_{uu}(s,u)\\&\qquad=\kappa'(s+u)\left\langle \mathbf{t}(s), \mathbf{n}(s+u)\right\rangle-\kappa(s+u)\,\tau(s+u)\left\langle \mathbf{t}(s), \mathbf{b}(s+u)\right\rangle.\end{aligned}$$

Evaluating at $u=0$, since $\phi(s,0)=0$, we obtain $\phi_u^2(s,0)=\kappa^2(s)$, so that $\phi_u(s,0)=\kappa(s)$. The third derivative is

$$\begin{aligned}&3\cos(\phi)\left(\kappa'(s+u)\,\kappa(s+u)-\phi_u\,\phi_{uu}\right)-\sin(\phi)\left(\kappa^2(s+u)\,\phi_u-\phi_u^3+\phi_{uuu}\right)\\&=\left(\kappa''(s+u)-\kappa(s+u)\,\tau^2(s+u)\right)\left\langle \mathbf{t}(s), \mathbf{n}(s+u)\right\rangle\\&-\left(2\,\kappa'(s+u)\,\tau(s+u)+\kappa(s+u)\,\tau'(s+u)\right)\left\langle \mathbf{t}(s), \mathbf{b}(s+u)\right\rangle.\end{aligned}$$

Evaluating at $u=0$, it reduces to

$$\kappa'(s)\,\kappa(s)-\phi_u(s,0)\,\phi_{uu}(s,0)=0.$$

Since $\phi_u(s,0)=\kappa(s)\neq 0$, we deduce $\phi_{uu}(s,0)=\kappa'(s)$.

Finally, the fourth derivative of (6.8) evaluated at $u=0$ gives

$$4\,\kappa(s)\,\kappa''(s)-4\,\kappa(s)\,\phi_{uuu}(s,0)=\kappa^2(s)\,\tau^2(s).$$

Since $\kappa(s)\neq 0$, we deduce

$$\phi_{uuu}(s,0)=\kappa''(s)-\frac{1}{4}\,\kappa(s)\,\tau^2(s).$$

□

Remark 6.13. From Lemma 6.12, notice that a third order approximation for the functions m and ϕ can be written:

$$m(s,L) = L + \frac{1}{24}\,\kappa^2(s)\,L^3 + \mathcal{O}\big(L^4\big).$$

and

$$\phi(s,u) = \kappa(s)\,u + \frac{\kappa'(s)}{2}\,u^2 + \frac{1}{6}\bigg(\kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s)\bigg)u^3 + \mathcal{O}\big(u^4\big).$$

Now an easy approximation for the area of a p-Holditch surface can be found in the general case using Lemma 6.12.

Theorem 6.14. *Let α be a regular closed curve with non-vanishing curvature κ. Given $p \in [0,1]$, if $A_H(p,L)$ is the area of the p-Holditch surface of α up to an admissible for regularity length $L > 0$, then*

$$\frac{A_H(p,L)}{p\,(1-p)} \approx \frac{L^2}{2}\int_I \kappa(s)\;\mathrm{d}s - \frac{L^4}{96}\int_I \kappa(s)\,\tau^2(s)\;\mathrm{d}s.$$

Proof. We are going to expand the double integral for the area of the considered p-Holditch surface,

$$F(L) = \int_I\int_0^{m(s,L)} \sin\phi(s,u)\;\mathrm{d}u\;\mathrm{d}s,$$

by Taylor series at $L = 0$. The desired expansion will have the form

$$F(L) = F(0) + L\,F'(0) + \frac{L^2}{2}\,F''(0) + \frac{L^3}{6}\,F'''(0) + \frac{L^4}{24}\,F^{(4)}(0) + \dots$$

First, note that $\phi(s,0) = 0$ and $m(s,0) = 0$, so $F(0) = 0$. Now, define

$$G_s(L) := \int_0^{m(s,L)} \sin\phi(s,u)\;\mathrm{d}u.$$

The first derivative of G_s is

$$G_s'(L) = \sin\phi\big(s, m(s,L)\big)\,m_L(s,L).$$

Since $G_s'(0) = 0$, we have that $F'(0) = 0$.

The second derivative of G_s is

$$\begin{aligned} G_s''(L) = {}& \cos\phi\big(s,m(s,L)\big)\,\phi_u\big(s,m(s,L)\big)\,m_L^2(s,L) \\ & + \sin\phi\big(s,m(s,L)\big)\,m_{LL}(s,L). \end{aligned}$$

Evaluating at $L = 0$, we find

$$G_s''(0) = \phi_u(s,0)\,m_L^2(s,0).$$

Since $m_L(s,0) = 1$ and $\phi_u(s,0) = \kappa(s)$ by Lemma 6.12, we have $G_s''(0) = \kappa(s)$, so that

$$F''(0) = \int_I \kappa(s)\,\mathrm{d}s.$$

The third derivative of G_s is

$$\begin{aligned} G_s'''(L) &= m_L\,\cos(\phi)\left(\phi_{uu}\,m_L^2 + 3\,\phi_u\,m_{LL}\right)\\ &\quad + \sin(\phi)\left(m_{LLL} - \phi_u^2\,m_L^3\right). \end{aligned}$$

Evaluated at $L = 0$ yields

$$G_s'''(0) = \phi_{uu}(s,0) + 3\,\kappa(s)\,m_{LL}(s,0).$$

Again, by Lemma 6.12, $\phi_{uu}(s,0) = \kappa'(s)$ and $m_{LL}(s,0) = 0$. Thus, $G_s''(0) = \kappa'(s)$ and

$$F'''(0) = \int_I \kappa'(s)\,\mathrm{d}s.$$

Since α is a closed curve, κ is periodic on I and we deduce $F'''(0) = 0$.

The fourth derivative of G_s evaluated at $L = 0$ is

$$G_s^{(4)}(0) = \left(\phi_{uuu}(s,0) + 3\,\kappa(s)\,m_{LLL}(s,0)\right) + \kappa(s)\left(m_{LLL}(s,0) - \kappa^2(s)\right).$$

By Lemma 6.12, $m_{LLL}(s,0) = \frac{1}{4}\,\kappa^2(s)$ and $\phi_{uuu}(s,0) = \kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s)$, so simplifying,

$$G_s^{(4)}(0) = \kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s).$$

Hence,

$$F^{(4)}(0) = -\frac{1}{4}\int_I \kappa(s)\,\tau^2(s)\,\mathrm{d}s,$$

where we have used that $\int_I \kappa''(s)\,\mathrm{d}s = 0$ because it is the integral of the derivative of a periodic function. $\square$

Remark 6.15. Following the same idea as in Lemma 6.12, it can be computed explicitly that

$$m_{LLLL}(s,0) = \kappa(s)\,\kappa'(s)$$

and

$$\phi_{uuuu}(s,0) = -\frac{1}{2}\,\tau^2(s)\,\kappa'(s) - \kappa(s)\,\tau(s)\,\tau'(s) + \kappa^{(3)}(s).$$

With that, also

$$G_s^{(5)}(0) = \frac{3}{2}\,\kappa^2(s)\,\kappa'(s) - \frac{1}{2}\tau^2(s)\,\kappa'(s) - \kappa(s)\,\tau(s)\,\tau'(s) + \kappa^{(3)}(s).$$

Now, notice that

$$-\frac{1}{2}\left(\kappa(s)\,\tau^2(s)\right)' = -\frac{1}{2}\tau^2(s)\,\kappa'(s) - \kappa(s)\,\tau(s)\,\tau'(s)$$

and

$$\frac{1}{2}\left(\kappa^3(s)\right)' = \frac{3}{2}\,\kappa^2(s)\,\kappa'(s).$$

Therefore, $G_s^{(5)}(0)$ can be written as a sum of three exact derivatives of periodic functions, which means that $F^{(5)}(0) = 0$ if α is closed. Hence, the expression given in Theorem 6.14 has 5th order of approximation:

$$\frac{A_H(p,L)}{p\,(1-p)} = \frac{L^2}{2}\int_I \kappa(s)\;\mathrm{d}s - \frac{L^4}{96}\int_I \kappa(s)\,\tau^2(s)\;\mathrm{d}s + \mathcal{O}(L^6).$$

Remark 6.16. The planar Holditch's theorem (also for non-closed curves) can be stated by means of the area swept out by a piece of the moving chord. In such a case, the Holditch area is equal to

$$\frac{1}{2}\,(p\,L)\,\big((1-p)\,L\big)\int_I \kappa(s)\;\mathrm{d}s,$$

where notice that the integral of the curvature measures the total angle swept out by the moving chord seen as an indicatrix (see [70] for a further discussion on this).

In this case, taking only the first term of the approximation given in Theorem 6.14, the exact value of the Holditch area is obtained.

Let's compute the Holditch area in an example.

Example 6.17. Consider the closed curve

$$\alpha(t) = \big(\cos t,\, \sin t,\, \sin^2 t\big), \quad t \in [0, 2\pi]\,.$$

The p-Holditch surface of α up to a length L is given by

$$\begin{aligned}\mathbf{x}^p(t,u) = \big(&p\cos(t+u) + (1-p)\cos(t),\, p\sin(t+u) + (1-p)\sin(t),\\ &p\sin^2(t+u) + (1-p)\sin^2(t)\big),\end{aligned}$$

for $t \in [0, 2\pi]$ and $u \in \,]0, m(t,L)[$. In Figure 6.4, two examples of these surfaces are represented for different choices of p. We have computed the function $m(t,L)$ numerically.

For example, the area of the Holditch surface for $L = 1$ and $p = 1/2$ computed as the area of a surface is $A_H(p,L) \approx 1.05461$. Using the expression of Theorem 6.14, we get 1.05497. If we cut the same expression to third order, the approximation is 1.06503. Using a higher approximation order, a more accurate answer is given. For instance, the value obtained for a sixth order approximation is 1.05453. See Figure 6.5 to compare, for each L, the value of Holditch area in the surface with the given approximations.

The maximum admissible length in this example is $L = 2$, for which $m(t,2) = \pi$ for all $t \in [0, 2\pi]$. The 1/2-Holditch curve of α for $L = 2$ is just a segment of double points. Moreover, the 1/2-Holditch surface of α up to $L = 2$ has, at the ends of that segment, two Whitney umbrella type singularities (see Figure 6.6).

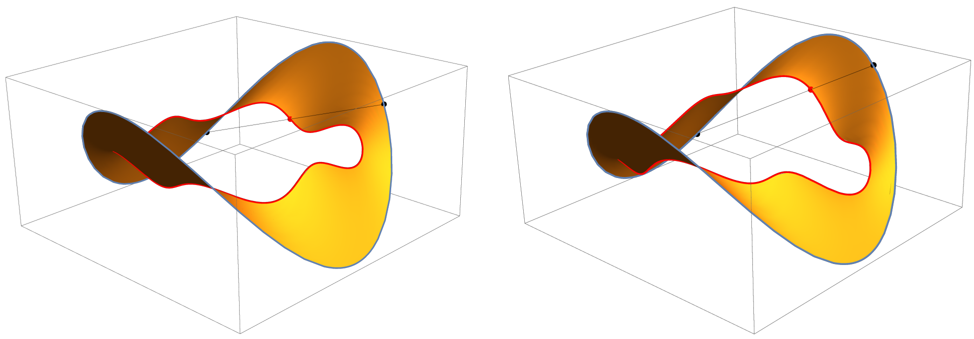

Figure 6.4: The p-Holditch surface of α up to a length $L = 3/2$. On the left, $p = 1/2$; on the right, $p = 1/3$.

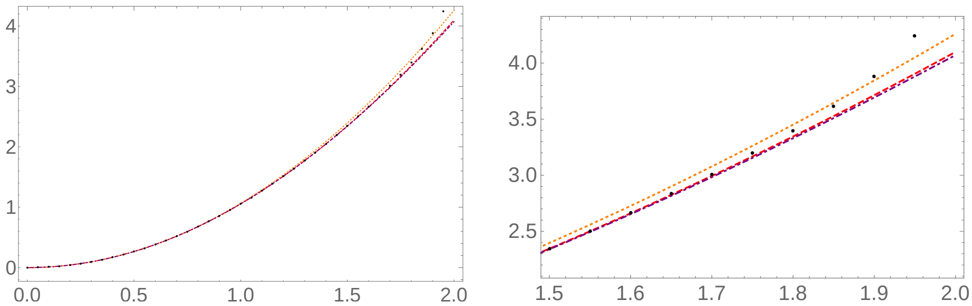

Figure 6.5: For $p = 1/2$, the big black points represent the shape of the Holditch area function when varying the length L of the chord. We represent in orange-dotted, the third order approximation function; in red-dashed, the fifth order and in purple-dot-dashed a sixth order approximation. For small values of L, they are very similar. For bigger L, little differences can be seen (right).

Let's present now another example of Holditch surface but for a non-closed space curve: the circular helix.

Example 6.18. Let $a > 0$, $b \neq 0$ and $c = \sqrt{a^2 + b^2}$. Consider as initial curve a circular helix parameterized by arc-length:

$$\alpha_{a,b}(s) = \left(a\cos\left(\frac{s}{c}\right),\, a\sin\left(\frac{s}{c}\right),\, \frac{b\,s}{c}\right).$$

In Figure 6.7, some p-Holditch surfaces of a circular helix are represented.

We know that $\kappa(s) = \frac{a}{a^2+b^2}$ and $\tau(s) = -\frac{b}{a^2+b^2}$ are constant. In this case,

$$\phi(s,u) = \arccos\left(\frac{b^2 + a^2\cos\left(\frac{u}{c}\right)}{c^2}\right).$$

By Proposition 6.6, the curvatures of the Holditch surfaces of $\alpha_{a,b}$ can be easily computed. The Gauss curvature is always negative:

$$K(s,u) = -\frac{b^2}{p\,q\left(a^2 + 2\,b^2 + a^2\cos\left(\frac{u}{c}\right)\right)^2}.$$

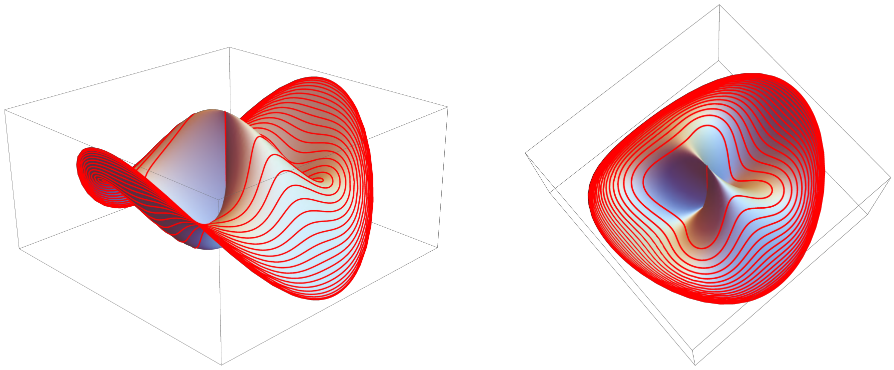

Figure 6.6: The 1/2-Holditch surface of α up to a length $L = 2$ exhibit two Whitney umbrella type singularities.

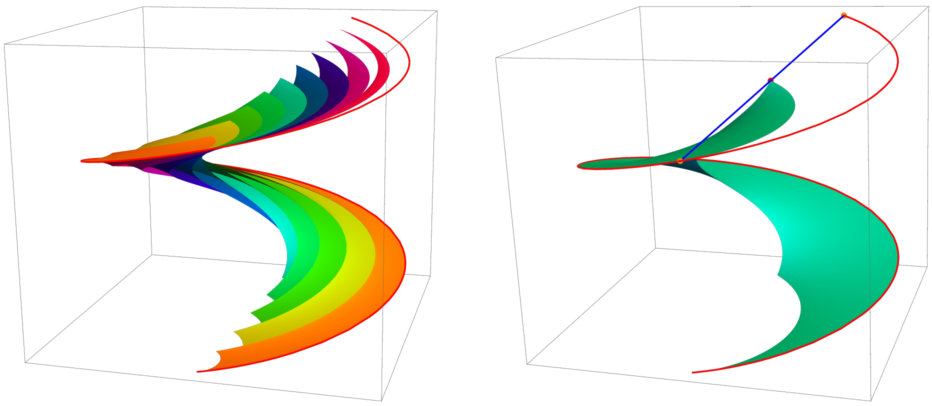

Figure 6.7: On the left, for different choices of p, some p-Holditch surfaces of $\alpha_{1,1/5}$ up to a length $L = 2$ are plotted. On the right, it is only represented the 1/2-Holditch surface, which is a helicoid.

And the mean curvature

$$H(s,u) = \frac{b\,c^2\,(q-p)\csc\!\left(\frac{u}{2c}\right)}{2\sqrt{2}\,a\,p\,q\left(a^2+2\,b^2+a^2\,\cos\!\left(\frac{u}{c}\right)\right)^2}$$

is zero if and only if $p = q = 1/2$.

Let us study separately the case $p = 1/2$. The parameterization of the 1/2-Holditch surface of $\alpha_{a,b}$ up to an admissible for regularity length L is

$$\mathbf{x}_{a,b}(s,u) = \Bigg(\frac{a}{2}\left(\cos\!\left(\frac{s}{c}\right)+\cos\!\left(\frac{s+u}{c}\right)\right), \frac{a}{2}\left(\sin\!\left(\frac{s}{c}\right)+\sin\!\left(\frac{s+u}{c}\right)\right), \frac{b\left(s+\frac{u}{2}\right)}{c}\Bigg).$$

We are going to prove that $\mathbf{x}_{a,b}$ is a helicoid. A helicoid has parameterization

$$\mathbf{y}_\beta(s,u) = \big(u\cos(s),\, u\sin(s),\, \beta s\big).$$

Since the parameter change

$$g_{a,b}(s,u) = \left(\frac{2\,s+u}{2\,c},\ a\cos\!\left(\frac{u}{2\,c}\right)\right)$$

verifies that

$$\mathbf{y}_b\big(g_{a,b}(s,u)\big) = \mathbf{x}_{a,b}(s,u),$$

then we have that the Holditch surface $\mathbf{x}_{a,b}$ is a helicoid.

Recall that minimal surfaces are those that locally minimize their areas. Equivalently, surfaces with zero mean curvature. Therefore, we have found that the helicoid is an example of minimal Holditch surface.

From the previous example, now we ask for all the minimal Holditch surfaces. It turns out that the unique non-planar regular minimal Holditch surface is the one seen above: the helicoid.

Theorem 6.19. *Let α be a regular curve with non-vanishing curvature and torsion. The only minimal non-planar regular Holditch surface is the helicoid.*

Proof. Using the local canonical form of α in a neighborhood of $u = 0$ up to third order (see [20], page 27), from Proposition 6.6, we obtain the first term of the Laurent series at $u = 0$ for the mean curvature H:

$$\frac{(p-q)\,\tau(s)}{4\,p\,q\,\kappa(s)}\,\frac{1}{u}.$$

If the Holditch surface is minimal, then $H = 0$. That implies $p = q = 1/2$. Thus, the Taylor series of H at $u = 0$ reduces to

$$\begin{aligned} H(s,u) = &-\frac{2\,\kappa'(s)\,\tau(s) + \kappa(s)\,\tau'(s)}{6\,\kappa^2(s)} \\ &+ \kappa'(s)\,\frac{4\,\kappa'(s)\,\tau(s) - \kappa(s)\,\tau'(s)}{12\,\kappa^3(s)}\,u + \mathcal{O}\big(u^2\big). \end{aligned}$$

Notice that it can be rewritten as

$$H(s,u) = -\frac{\big(\kappa^2(s)\,\tau(s)\big)'}{6\,\kappa^3(s)} + \frac{\kappa'(s)\,\tau^2(s)}{12\,\kappa^6(s)}\left(\frac{\kappa^4(s)}{\tau(s)}\right)'u + \mathcal{O}\big(u^2\big).$$

Because of being $H = 0$, from the term in u we deduce that one, $\kappa(s)$ or $\kappa^4(s)/\tau(s)$, is constant. In the first case, if κ is constant, from the independent term, τ must be constant. In the second case, if $\kappa^4(s)/\tau(s)$ is constant, since the independent term implies $\kappa^2(s)\,\tau(s)$ constant, the same conclusion is deduced: both κ and τ are constant. Therefore α must be a circular helix.

Since we have seen in the Example 6.18 that the 1/2-Holditch surface of a circular helix is a helicoid, which is minimal, the rest of the terms in the Taylor series of H will be automatically zero and satisfied with the condition of being κ and τ constant. $\square$

Remark 6.20. From the example of the circular helix, one can ask if the generated Holditch surfaces (or the helicoid, in particular) can be extended to the "outside" part of the cylinder which contains the helix. In other words, is there any family of surfaces naturally defined such that connect well with the initial curve and the defined Holditch surfaces? The answer to this question is affirmative but it may be worth a detailed full study in a future work.

Conclusions

As the reader may have noticed after reading this work, one of the objectives was to make this thesis a self-contained reference for anyone who wants to be introduced to Holditch's theorem and related topics.

At the beginning, after having understood some proofs of the classical Holditch theorem, the first question we tried to solve was the one which was answered in our paper [65]: where is the hidden ellipse of Holditch's statement? The first step was dealing with some examples, namely closed polygonal curves, where the hidden ellipse could be explicitly seen. Some theoretical results on the existence of Holditch curves and the continuity of the map which sends any simple closed curve into its Holditch curve for a chord length were also deduced.

The number of articles on Holditch's theorem we found in a first bibliographic search was quite small. The main references were the articles by Broman ([11] and [10]) and Cooker ([13] and [14]). Since these works considered the planar case, it seemed that the next natural step was to ask for a Holditch-type theorem in other 2-dimensional geometries. Nevertheless, after having obtained our own proof for a Holditch statement in the sphere, we became aware of the existence of some old papers written and published in Spanish where the same result was given (see [91], [95] and [83]). Some of these papers were reprinted on the occasion of the publication of the complete or selected works of the authors, but with not too much diffusion.

In this setting, we tried to understand the techniques used by Santaló in non-Euclidean geometry. However, instead of using differential forms to achieve the Holditch result, we found another direct proof by relating the geodesic curvatures of the initial curve and the Holditch one. In fact, this result could be extended for the general kind of curves considered in this work under the name of *generated curves*. Thus, as particular cases, the parallel curves, constant width curves and bicycle curves appeared naturally, together with their associated famous results on areas and lengths due to Steiner and Barbier. This new procedure gave rise to our preprint [64].

Once the Holditch statement for constant curvature surfaces was completed, we addressed the following question: which is the closed curve in the constant curvature manifold that plays the role of the planar hidden ellipse? With the analogue in the constant curvature surface of the planar da Vinci construction of an ellipse we found the desired curve as the intersection of the surface with a quartic curve.

Moreover, we found that Jacobi fields along the moving chord were an important tool to understand the behavior of such chords and to manage overlapping area problems.

Another interesting question was to find an analogous result to Holditch for space curves. The main difficulty was to know from which surface one had to compute the Holditch area. At first, looking at some simple examples we thought of the area swept out by the moving chord or of some natural constructions with ruled surfaces. None of these led us to a statement in the desired form. Finally, after many tries, the correct surface to compute areas was found (the Holditch surface) thanks to an idea which came from the paper [2] by Arnol'd. This generalization of Holditch's theorem constituted our paper [66].

After finishing the writing of this thesis, we think that there are still interesting questions to answer in a future work. Among others, we point out the following ideas:

1. To find a Holditch type statement or some bounds on areas in any 2-dimensional manifold, not necessarily of constant curvature.

2. For space curves, to define naturally a family of surfaces that connect well with the defined Holditch surfaces and to study some of their properties and relations within areas. Also, given some properties of the initial curve (e.g. regarding its curvature or torsion), to find which properties its Holditch surface must satisfy.

3. To study the retrograde motion phenomenon on Holditch's theorem in M^K.

4. To generalize da Vinci's construction of an ellipse to any 2-manifold and to study the properties of constructed curves.

5. To relate the concept of ambiguous curves in closed bicycle tire-tracks with Holditch curves.

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